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# Conservatism and Liquidity Traps

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# Conservatism and Liquidity Traps*

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**Abstract**

In an economy with an occasionally binding zero lower bound (ZLB) constraint, the *anticipation* of future ZLB episodes creates a trade-off for discretionary central banks between inflation and output stabilization. As a consequence, inflation systematically falls below target even when the policy rate is above zero. Appointing Rogoff's (1985) conservative central banker mitigates this deflationary bias away from the ZLB and enhances welfare by improving allocations both at and away from the ZLB.

*Keywords:* Deflationary Bias, Inflation Conservatism, Inflation Targeting, Liquidity Traps, Zero Lower Bound
*JEL-Codes:* E52, E61

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# 1 Introduction

Over the past few decades, a growing number of central banks around the world have adopted inflation targeting as a policy framework. The performance of inflation targeting in practice has been widely considered a success.[1] However, some economists and policymakers have voiced the need to re-examine central banks' monetary policy frameworks in light of the liquidity trap conditions currently prevailing in many advanced economies.[2] As shown in Eggertsson and Woodford (2003) among others, the zero lower bound (ZLB) on nominal interest rates severely limits the ability of inflation-targeting central banks to stabilize the economy absent an explicit commitment technology.[3] Some argue that the ZLB is likely to bind more frequently and that liquidity trap episodes might hit the economy more severely in the future than they have in the past.[4] Understanding the implications of the ZLB for the conduct of monetary policy is therefore of the utmost importance for economists and policymakers alike.

In this paper, we contribute to this task by examining the desirability of Rogoff's (1985) conservative central banker in a standard New Keynesian model in which large contractionary shocks occasionally push the policy rate to the ZLB. Rogoff (1985) showed that in a model where a lack of commitment leads to an inflation bias, society can be better off if the central bank is less concerned with output gap stability relative to inflation stability than is society. To focus on the role of the ZLB, we abstract from the original inflation bias by assuming that the steady-state distortions are eliminated by appropriate subsidies. Society's welfare is then given by the negative of the weighted sum of inflation and output volatility. We analyze how the economy behaves under a discretionary central banker with an alternative weight on output volatility, and we compute the optimal weight that maximizes society's welfare.

We find that the appointment of a fully conservative central banker—that is, a banker who places zero weight on output stabilization—is optimal in our baseline model, which features only a demand shock. That is, society's welfare is maximized when the central bank focuses exclusively on inflation stabilization. The mechanism behind our result is as follows. In the economy in which future shocks can push the policy rate to the lower bound, the anticipation of lower inflation and output gives forward-looking households and firms incentives to reduce consumption and prices even when the policy rate is above the ZLB. The central bank cannot fully counteract these incentives. When the central bank is concerned with both inflation and output stabilization, it faces a trade-off between the two objectives, implying deflation and a positive output gap in those states where the ZLB is not binding. Following the terminology of Nakov (2008), we will refer to this deflation when the policy rate is above zero as *deflationary bias*.

A central banker who puts comparatively more weight on inflation stabilization mitigates the deflation bias away from the ZLB at the cost of a potentially higher output gap. Viewed in isolation,

[1] See, for instance, Walsh (2009) and Svensson (2010), and the references therein.
[2] See, for example, Blanchard, Dell'ariccia, and Mauro (2010); Tabellini (2014); Williams (2014).
[3] See also Jung, Teranishi, and Watanabe (2005), Adam and Billi (2007), and Nakov (2008).
[4] See, for example, IMF (2014) and Chung, Laforte, Reifschneider, and Williams (2012).

this is welfare-reducing because it shifts inflation and output gap realizations away from the welfare-implied target criteria. However, lower deflation and higher output gaps away from the ZLB also reduce expected real interest rates and increase the expected marginal costs at the ZLB, mitigating deflation and output declines there. This in turn allows the central bank to achieve zero inflation with a smaller positive output gap away from the ZLB, setting in motion a positive feedback loop. We prove analytically the optimality of placing zero weight on output stabilization for the baseline version of our model in which the demand shock follows a two-state Markov process, and we confirm this result numerically for a version of the model with a first order autoregressive shock process.

The desirability of conservatism is robust to introducing cost-push shocks into the economy, but the optimal weight on output stabilization may no longer be zero. In the model with demand shocks and cost-push shocks, the optimal weight would coincide with society's weight in the absence of the ZLB. Accounting for the ZLB, the optimal weight lies between zero and society's weight, as long as the cost-push shock is sufficiently small and the demand shock is the key driver of liquidity trap episodes. The greater the frequency of the ZLB episodes, the closer the optimal weight is to zero. This observation makes intuitive sense and is reminiscent of the finding in Coibion, Gorodnichenko, and Wieland (2012) that the effect of the ZLB on the optimal inflation target is larger when the ZLB constraint binds more frequently.

Our result may initially strike some readers as counterintuitive.[5] The desirability of assigning a higher weight on the inflation objective was shown originally in a framework in which the lack of an explicit commitment technology leads to inflation that is too high. The problem of the economy facing the ZLB constraint is the opposite—inflation that is too low—which may lead one to conjecture that the opposite prescription of assigning a *lower* weight on inflation would be desirable.[6] Our analyses show that this is not the case. In describing why, we trace out the beneficial effects of stabilizing inflation expectations, which are central to understanding the desirability of the conservative central bank in the original model of Rogoff (1985) as well.

A valuable byproduct of our analysis of conservatism is a closed-form characterization of the conditions that guarantee the existence of the standard Markov-Perfect equilibrium with occasionally binding ZLB constraints. Some researchers have reported difficulty in obtaining numerical convergence when solving the model with the ZLB and have suggested that equilibrium under some parameter configurations does not exist.[7] Yet not much is known about the conditions for equilibrium existence. We prove that the equilibrium ceases to exist when the frequency and persistence of crisis shocks are sufficiently high, and we provide analytical expressions for the frequency and persistence thresholds at which this occurs. This result should be a useful reference for those who numerically solve the New Keynesian model with occasionally binding ZLB constraints.

[5] In the economic debate, monetary conservatism is sometimes equalized with a preference for lower inflation. In contrast, Rogoff's definition of monetary conservatism that we adopt here concerns the weight that a central bank with a symmetric loss function puts on deviations of inflation from target relative to its other objectives.

[6] Tabellini (2014), for example, conjectures that a lower weight on the output stability objective is detrimental to stabilization policy in the model that incorporates the ZLB constraint.

[7] See, for example, Adam and Billi (2007) and Billi (2013) for the non-convergence result under the Markov-Perfect equilibrium and Richter and Throckmorton (2014) under the Taylor-rule equilibrium.

While our analysis focuses on the standard Markov-Perfect equilibrium that fluctuates around a positive nominal interest rate so that the ZLB constraint binds only occasionally, there exists a second Markov-Perfect equilibrium in which the nominal interest rate is at zero permanently. In the Appendix, we also provide analytical characterizations of the conditions for the existence of this deflationary Markov-Perfect equilibrium.

Our analysis is distinct from the majority of the literature on liquidity traps as we focus on the implications of *anticipation* of future ZLB episodes for the policy trade-offs and stabilization performance when the policy rate is *away from* the ZLB. Many prominent papers in the literature assume that there exists an absorbing state in which the ZLB does not bind. That is, once the economy is out of the liquidity trap, the private sector agents attach zero probability to the event that future shocks will push the policy rate back to the ZLB.[8] Some recent papers have analyzed various issues related to the ZLB in a fully stochastic framework where the agents are aware of the possibility of returning to the ZLB, but none of them has analyzed the implication of the ZLB risk for the design of the central bank's objective.

Our paper is related to a set of papers that have examined various ways to improve allocations at the ZLB in time-consistent manners. Eggertsson (2006), Burgert and Schmidt (2014), and Bhattarai, Eggertsson, and Gafarov (2014) considered economies in which the government can choose the level of nominal debt and showed that an increase in government bonds during the liquidity trap improves allocations by creating incentives for future governments to inflate. In a model in which government spending is valued by the household, Nakata (2013) and Schmidt (2013) showed that a temporary increase in government spending can improve welfare whenever the policy rate is stuck at the ZLB.[9] A key characteristic of these proposals is that they involve additional policy instruments and require perfect coordination of monetary and fiscal authorities. The approach studied in our paper only requires that the central bank is maximizing its assigned objective.

A few recent papers examine other time-consistent ways to better stabilize inflation and output in the model with the ZLB constraint without relying on additional policy instruments. Nakata (2014) demonstrates that a reputational concern on the part of the central bank can make the promise of overshooting inflation and output time-consistent. Billi (2013) revisits the desirability of assigning a nominal-income stabilization objective to the central bank. In our ongoing work, we compare the relative benefits of various alternative objectives, including interest-rate smoothing, price-level stabilization, and nominal-income stabilization (Nakata and Schmidt, 2015).

This paper is also related to a set of papers that examine the desirability of Rogoff's conservative central banker in settings other than the original model with inflation bias. Clarida, Gali, and Gertler (1999) showed that the appointment of a conservative central banker is also desirable in a New Keynesian model, in which the presence of persistent cost-push shocks creates a stabilization

[8]See Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011) among others.

[9]Schmidt (2014) examines what type of *fiscal* policymaker is best suited for dealing with liquidity traps in the absence of policy commitment. He finds that an activist fiscal authority that cares less about government consumption stability relative to output gap and inflation stability than society does is welfare-improving.

bias in discretionary monetary policy—that is, an inferior short-run trade-off between inflation and output stabilization compared with the time-inconsistent Ramsey policy. Adam and Billi (2008), Adam and Billi (2014), and Niemann (2011) examined the benefit of conservatism in versions of New Keynesian models augmented with endogenous fiscal policy. However, all of these authors have abstracted from the ZLB constraint.

Finally, our analyses of the conditions that guarantee the existence of standard and deflationary Markov-Perfect equilibria are related to the analyses by Tambakis (2014) and Armenter (2014). Tambakis (2014) characterizes the conditions that guarantee the existence of the standard Markov-Perfect equilibrium while assuming that the probability of the crisis shock does not depend on the state of the economy. We extend his results by considering both standard and deflationary Markov-Perfect equilibria and allowing for state dependence in the distribution of shocks. Armenter (2014) shows that the deflationary Markov-Perfect equilibrium exists in an economy with $n$-state Markov shocks if and only if the standard Markov-Perfect equilibrium exists, but he is silent about the conditions under which these equilibria exist. We focus on an economy with two-state Markov shocks and provide a complete characterization of the conditions for the existence of both types of Markov-Perfect equilibria.[10]

The rest of the paper is organized as follows. Section 2 describes the model and the government's optimization problem, and defines the welfare measure. Section 3 presents the main results. Section 4 extends the analysis to a model with both demand and cost-push shocks and to a continuous-state model. After a brief discussion in Section 5 on how our analyses relate to the literature on the optimal inflation target, Section 6 concludes.

# 2 The model

This section presents the model, lays down the policy problem of the central bank and defines the equilibrium.

## 2.1 Private sector

The private sector of the economy is given by the standard New Keynesian structure formulated in discrete time with infinite horizon as developed in detail in Woodford (2003) and Gali (2008). A continuum of identical, infinitely-living households consumes a basket of differentiated goods and supplies labor in a perfectly competitive labor market. The consumption goods are produced by

[10] Several studies have characterized the conditions for the existence of Taylor-rule equilibria in models with the ZLB. Eggertsson (2011) and Braun, Körber, and Waki (2013) characterize the conditions guaranteeing the existence of a Taylor-rule equilibrium in a semi-loglinear model, assuming that the economy eventually reverts back to an absorbing state and the ZLB does not bind in the absorbing state. Christiano and Eichenbaum (2012) analyzed the existence and multiplicity of Taylor-rule equilibria in a fully nonlinear model, again assuming the eventual return to the steady-state where the ZLB does not bind. Mendes (2011) characterizes the conditions for the existence of the standard and deflationary Taylor-rule equilibria in a fully stochastic semi-loglinear New Keynesian economy, assuming that the process for the natural rate of interest has no persistence. In an early contribution, Benhabib, Schmitt-Grohe, and Uribe (2001) show the existence of two *steady-states* in a sticky-price economy that abstracts from fundamental shocks.

firms using (industry-specific) labor. Firms maximize profits subject to staggered price-setting as in Calvo (1983). Following the majority of the literature on the ZLB, we put all model equations except for the ZLB constraint in semi-loglinear form. This allows us to derive closed-form results.

The equilibrium conditions of the private sector are given by the following two equations:

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} \tag{1}$$

and

$$y_t = E_t y_{t+1} - \sigma \left( i_t - E_t \pi_{t+1} - r^* \right) + d_t. \tag{2}$$

where $\pi_t$ is the inflation rate between period $t-1$ and $t$, $y_t$ denotes the output gap, $i_t$ is the level of the nominal interest rate between period $t$ and $t+1$, and $d_t$ is an exogenous demand shock capturing fluctuations in the natural real rate of interest, $r_t := r^* + \frac{1}{\sigma} d_t$. Equation (1) is a standard New Keynesian Phillips curve and equation (2) is the consumption Euler equation. The parameters are defined as follows: $\beta \in (0,1)$ denotes the representative household's subjective discount factor, $\sigma > 0$ is the intertemporal elasticity of substitution in consumption, and $r^* = \frac{1}{\beta} - 1$ is the deterministic steady state of the natural real rate. $\kappa$ represents the slope of the New Keynesian Phillips curve and is related to the structural parameters of the economy as follows:

$$\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\eta\theta)} \left( \sigma^{-1} + \eta \right), \tag{3}$$

where $\alpha \in (0,1)$ denotes the share of firms that cannot reoptimize their price in a given period, $\eta > 0$ is the inverse of the elasticity of labor supply, and $\theta > 1$ denotes the price elasticity of demand for differentiated goods.

We assume that the demand shock $d_t$ follows a two-state Markov process, which allows us to reveal the underlying mechanism in a simple and intuitive way. In particular, $d_t$ takes the value of either $d_H$ or $d_L$ where we refer to $d_H > -\sigma r^*$ as the high state and $d_L < -\sigma r^*$ as the low state. The transition probabilities are given by

$$\text{Prob}(d_{t+1} = d_L | d_t = d_H) = p_H \tag{4}$$

and

$$\text{Prob}(d_{t+1} = d_L | d_t = d_L) = p_L. \tag{5}$$

$p_H$ is the probability of moving to the low state in the next period when the economy is in the high state today and will be referred to as the *frequency* of the contractionary shocks. $p_L$ is the probability of staying in the low state when the economy is in the low state today and will be referred to as the *persistence* of the contractionary shocks. We will also refer to high and low states as non-crisis and crisis states, respectively. Unlike the majority of the papers which adopt this two-state shock framework, we do not assume that the non-crisis state is an absorbing state, i.e. $p_H = 0$. As a result, even when the policy rate is away from the ZLB, there is a positive probability that the ZLB binds in the future. This will be a key force in our model.

In Section 4, we extend the analysis to a continuous-state model in which the demand shock follows a stationary autoregressive process.

## 2.2 Society's objective and the central bank's problem

We assume that society's value, or welfare, at time $t$ is given by the expected discounted sum of future utility flows,

$$V_t = u(\pi_t, y_t) + \beta E_t V_{t+1}, \tag{6}$$

where society's contemporaneous utility function, $u(\cdot, \cdot)$, is given by the standard quadratic function of inflation and the output gap,

$$u(\pi, y) = -\frac{1}{2}\left(\pi^2 + \bar{\lambda} y^2\right). \tag{7}$$

This objective function can be motivated by a second-order approximation to the household's preferences. In such a case, $\bar{\lambda}$ is a function of the structural parameters and is given by $\bar{\lambda} = \frac{\kappa}{\theta}$.

The value for the central bank is given by

$$V_t^{CB} = u^{CB}(\pi_t, y_t) + \beta E_t V_{t+1}^{CB}, \tag{8}$$

where the central bank's contemporaneous utility function, $u^{CB}(\cdot, \cdot)$, is given by

$$u^{CB}(\pi, y) = -\frac{1}{2}\left(\pi^2 + \lambda y^2\right). \tag{9}$$

Note that, while the central bank's objective function resembles the private sector's, the relative weight that it attaches to the stabilization of the output gap, $\lambda \geq 0$, may differ from $\bar{\lambda}$. We assume that the central bank does not have a commitment technology. Each period $t$, the central bank chooses the inflation rate, the output gap, and the nominal interest rate in order to maximize its objective function subject to the behavioral constraints of the private sector, with the policy functions at time $t+1$ taken as given. The problem of the central bank is thus given by

$$V_t^{CB}(d_t) = \max_{\pi_t, y_t, i_t} \quad u^{CB}(\pi_t, y_t) + \beta E_t V_{t+1}^{CB}(d_{t+1}) \tag{10}$$

subject to the zero lower bound constraint,

$$i_t \geq 0, \tag{11}$$

and the private-sector equilibrium conditions (1) and (2) described above.

A *Markov-Perfect equilibrium* is defined as a set of time-invariant value and policy functions $\{V^{CB}(\cdot),\ y(\cdot),\ \pi(\cdot),\ i(\cdot)\}$ that solves the central bank's problem above, together with society's value function $V(\cdot)$, which is consistent with $y(\cdot)$ and $\pi(\cdot)$. As discussed in Armenter (2014) and Nakata (2014), there are two Markov-Perfect equilibria in this economy: One fluctuates around a positive nominal interest rate and zero inflation/output (*the standard Markov-Perfect equilibrium*), and the other fluctuates around a zero nominal interest rate and negative inflation/output (*the deflationary Markov-Perfect equilibrium*). While the deflationary Markov-Perfect equilibrium is interesting, we focus on the standard Markov-Perfect equilibrium in this paper. In most economies that have recently faced a liquidity trap, long-run inflation expectations have been well anchored to some positive rate and various survey data strongly suggests that private-sector agents expect the central bank to eventually raise the policy rate. Thus, the standard Markov-Perfect equilibrium seems to be more relevant on empirical grounds.[11]

The main exercise of the paper will be to examine the effects of $\lambda$ on welfare. We quantify the welfare of an economy by the perpetual consumption transfer (as a share of its steady state) that would make a household in the economy indifferent to living in the economy without any fluctuations. This is given by

$$W := (1-\beta)\,\frac{\theta}{\kappa}\left(\sigma^{-1}+\eta\right)E[V], \tag{12}$$

where the mathematical expectation is taken with respect to the unconditional distribution of $d_t$.

## 3 Results

After providing conditions for the existence of the standard Markov-Perfect equilibrium, this section shows how output and inflation in the two states depend on the central bank's relative weight on output stabilization $\lambda$ and shows that $\lambda = 0$ is optimal. The second part of this section provides a numerical illustration.

The standard Markov-Perfect equilibrium is given by a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the following system of linear equations—

$$y_H = \left[(1-p_H)y_H + p_H y_L\right] + \sigma\left[(1-p_H)\pi_H + p_H\pi_L - i_H + r^*\right] + d_H, \tag{13}$$

$$\pi_H = \kappa y_H + \beta\left[(1-p_H)\pi_H + p_H\pi_L\right], \tag{14}$$

$$0 = \lambda y_H + \kappa\pi_H, \tag{15}$$

$$y_L = \left[(1-p_L)y_H + p_L y_L\right] + \sigma\left[(1-p_L)\pi_H + p_L\pi_L - i_L + r^*\right] + d_L, \tag{16}$$

$$\pi_L = \kappa y_L + \beta\left[(1-p_L)\pi_H + p_L\pi_L\right], \tag{17}$$

and

$$i_L = 0, \tag{18}$$

—and satisfies the following two inequality constraints:

$$i_H > 0 \tag{19}$$

[11] An exception is Japan where the policy rate has been at the ZLB for more than a decade.

and

$$\lambda y_L + \kappa \pi_L < 0. \tag{20}$$

For any variable $x$, $x_k$ denotes the value of that variable in the $k$ state where $k \in \{H, L\}$. The first inequality constraint checks the nonnegativity of the nominal interest rate in the high state, and the second checks nonpositivity of the Lagrangean multiplier on the ZLB constraint in the low state.

The model can be solved in closed form. We first prove key properties of the model and then move on to numerical analyses.

## 3.1 Analytical results

**Proposition 1**: The standard Markov-Perfect equilibrium exists if and only if

$$p_L \leq p_L^*(\Theta_{(-p_L)})$$

and

$$p_H \leq p_H^*(\Theta_{(-p_H)}),$$

where i) for any parameter $x$, $\Theta_{(-x)}$ denotes the set of parameter values excluding $x$, and ii) the cutoff values $p_L^*(\Theta_{(-p_L)})$ and $p_H^*(\Theta_{(-p_H)})$ are given in Appendix A.

See Appendix A for the proof. The two conditions guarantee the nonpositivity of the Lagrange multiplier in the crisis state and the nonnegativity of the nominal interest rate in the non-crisis state, respectively. When the frequency of the contractionary shock, $p_H$, is high, the central bank reduces the nominal interest rate aggressively to mitigate the deflation bias, which will be described shortly. Thus, for the policy rate to be positive in the high state, $p_H$ must be sufficiently low. With $p_L > p_L^*(\Theta_{(-p_L)})$, inflation and output in the low state are positive when they satisfy the consumption Euler equation and the Phillips curve. Though this is somewhat unintuitive, it makes sense. When the persistence of the crisis, $p_L$, is high, inflation and output in today's low state are largely dependent on households' and firms' expectations of inflation and output in the next period's low state. Thus, positive inflation and output in the low state can be self-fulfilling. However, such positive inflation and output cannot be an equilibrium because the central bank would have incentives to raise the nominal interest rate from zero in the low state. This incentive manifests itself in the positive Lagrangean multiplier in the low state when inflation and output are positive.[12]

When the conditions for the existence of the equilibrium hold, the signs of the endogenous variables are unambiguously determined.

[12] The conditions for the existence of the other Markov-Perfect equilibrium turn out to be identical to those for the existence of the standard Markov-Perfect equilibrium; see Appendix E.

**Proposition 2**: For any $\lambda \geq 0$, $\pi_H \leq 0$, $y_H > 0$, $i_H < r_H$, $\pi_L < 0$, and $y_L < 0$. With $\lambda = 0$, $\pi_H = 0$.

See Appendix A for the proof. In the low state, the ZLB constraint becomes binding, and output and inflation are below target. In the high state, a positive probability of entering the low state in the next period reduces expected marginal costs of production and leads firms to lower prices in anticipation of future crises events. This raises the expected real rate faced by the representative household and gives it an incentive to postpone consumption. The central bank chooses to lower the nominal interest rate to mitigate these anticipation effects. In equilibrium, inflation and output in the high state are negative and positive, respectively, and the non-crisis policy rate is below the natural real interest rate. These analytical results are consistent with the numerical results in the literature (see Nakov (2008), among others). In particular, negative inflation away from the ZLB has been referred to as *deflationary bias*. This proposition provides the first analytical underpinning for the deflation bias.

Notice that the first part of this proposition ($\pi_H \leq 0$, $y_H > 0$, and $i_H < r_H$) can be seen as demonstrating the breakdown of the so-called divine coincidence. If there were no ZLB constraint, then inflation and output gap in both states would be zero. Here, in the model with the ZLB constraint, inflation and output gap are not fully stabilized even in the high state when the ZLB does not bind. This is because the possibility of future ZLB episodes reduces inflation expectations in the high state, which can be thought of as a negative cost-push shock that shifts down the intercept of the Phillips curve. In this regard, accounting for $p_H > 0$ is essential for the analysis. Imposing $p_H = 0$, that is, assuming that the non-crisis state is an absorbing state, as is often done in the literature to simplify the analysis, is not innocuous as it buries both the deflation bias and the breakdown of the divine coincidence away from the ZLB.

We now establish several results on how the degree of conservatism affects endogenous variables in both states. In doing so, we assume that parameter values are such that the conditions for equilibrium hold for a reasonable range of $\lambda > 0$.[13]

**Proposition 3**: For any $\lambda \geq 0$, $\frac{\partial \pi_H}{\partial \lambda} < 0$, $\frac{\partial \pi_L}{\partial \lambda} < 0$, and $\frac{\partial y_L}{\partial \lambda} < 0$. For any $\lambda \geq 0$, $\frac{\partial y_H}{\partial \lambda} < 0$ if and only if $\beta p_H - (1-\beta)\left(\frac{1-p_L}{\kappa\sigma}(1-\beta p_L + \beta p_H) - p_L\right) < 0$.

See Appendix A for the proof. $\frac{\partial \pi_H}{\partial \lambda} < 0$ means that, as the central bank cares more about inflation, inflation in the high state is higher (i.e., the deflation bias in the high state is smaller). Since a lower rate of deflation in the high state increases output and inflation in the low state via expectations, inflation and output in the low state both increase with the degree of conservatism ($\frac{\partial \pi_L}{\partial \lambda} < 0$ and $\frac{\partial y_L}{\partial \lambda} < 0$). The effect of conservatism on output in the high state is ambiguous. On the one hand, a more conservative central bank is willing to tolerate a larger overshooting of output given the same inflation expectations. On the other hand, higher inflation in both states improves the trade-off between inflation and output stabilization implied by the Phillips curve, making it possi-

[13] $p_H^*$ and $p_L^*$ do depend on $\lambda$, but for values of $\lambda$ between 0 and $\bar{\lambda}$ the quantitative effect is negligible.

ble to reduce the overshooting of output in the non-crisis state. Proposition 3 demonstrates that the former effect dominates the latter if and only if $\beta p_H - (1-\beta)\left(\frac{1-p_L}{\kappa\sigma}(1-\beta p_L+\beta p_H)-p_L\right) < 0$.

**Proposition 4**: Suppose that $p_L$ and $p_H$ are sufficiently low so that $p_L \leq p_L^*(\Theta_{(-p_L)})$ and $p_H \leq p_H^*(\Theta_{(-p_H)})$ for all $\lambda$ in $[0,\bar{\lambda}]$. Then, welfare is maximized at $\lambda = 0$.

See Appendix A for the proof. As demonstrated in Proposition 3, deflation in the high state is smaller and inflation and output decline less in the low state with a smaller $\lambda$. These forces work to improve society's welfare. If $\beta p_H - (1-\beta)\left(\frac{1-p_L}{\kappa\sigma}(1-\beta p_L+\beta p_H)-p_L\right) > 0$, then output in the high state becomes smaller with a smaller $\lambda$ and the optimality of zero weight is obvious. If $\beta p_H - (1-\beta)\left(\frac{1-p_L}{\kappa\sigma}(1-\beta p_L+\beta p_H)-p_L\right) < 0$, then a smaller $\lambda$ increases the already positive output gap and thus has ambiguous effects on welfare. Proposition 4 demonstrates that, even in this case, the beneficial effects of a smaller $\lambda$ on $\pi_H$, $\pi_L$, and $y_L$ dominate the adverse effect on $y_H$.[14]

### 3.2 Numerical illustration

Table 1: Parameterization (Two-state shock model)

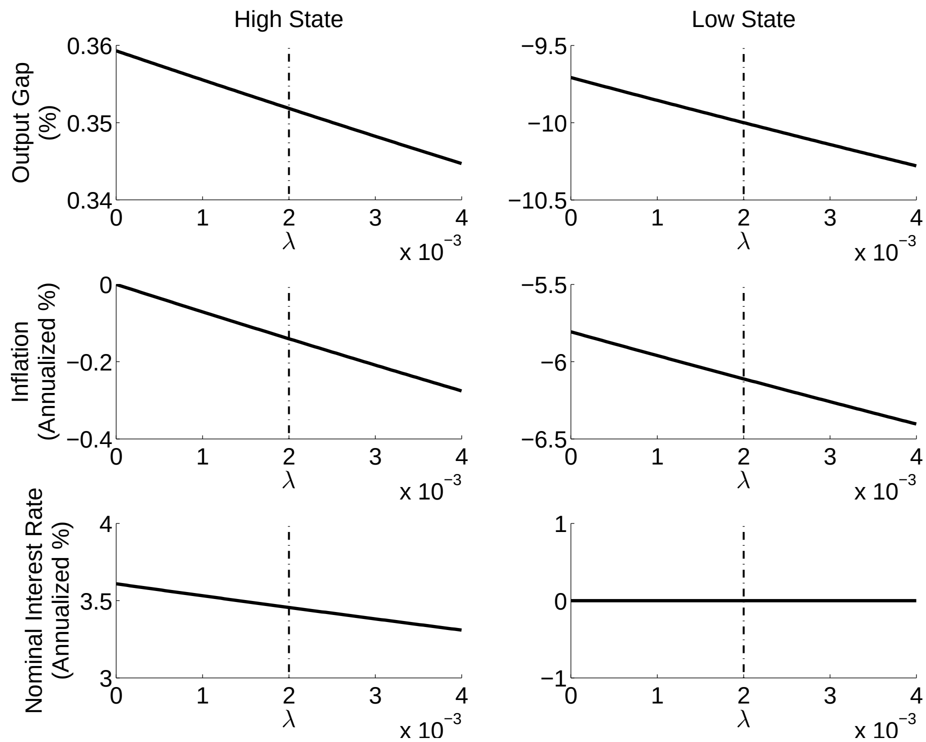
| Parameter | Value | Economic interpretation |
|---|---|---|
| $\beta$ | 0.99 | Subjective discount factor |
| $\sigma$ | 0.5 | Intertemporal elasticity of substitution in consumption |
| $\eta$ | 0.47 | Inverse labor supply elasticity |
| $\theta$ | 10 | Price elasticity of demand |
| $\alpha$ | 0.8106 | Share of firms per period keeping prices unchanged |
| $d_H$ | 0 | Demand shock in the high state |
| $d_L$ | -0.0113 | Demand shock in the low state |
| $p_H$ | 0.005 | Frequency of contractionary demand shock |
| $p_L$ | 0.875 | Persistence of contractionary demand shock |

We now illustrate the aforementioned properties of the model with specific parameter values. The structural parameters are calibrated using the parameter values from Eggertsson and Woodford (2003), as listed in Table 1. The frequency of the crisis shock is chosen so that the ZLB episode occurs once per five decades, on average. In the majority of the papers who have adopted this framework, $p_H$ is assumed to be zero. The persistence of 0.875 means that the expected duration of the crisis is two years. The size of the shock is chosen so that the decline in output during the crisis is 10 percent.

Figure 1 shows how the output gap, inflation, and the nominal interest rate in both states vary with the weight on output stabilization, $\lambda$. The dashed vertical lines show society's weight, $\bar{\lambda}$.

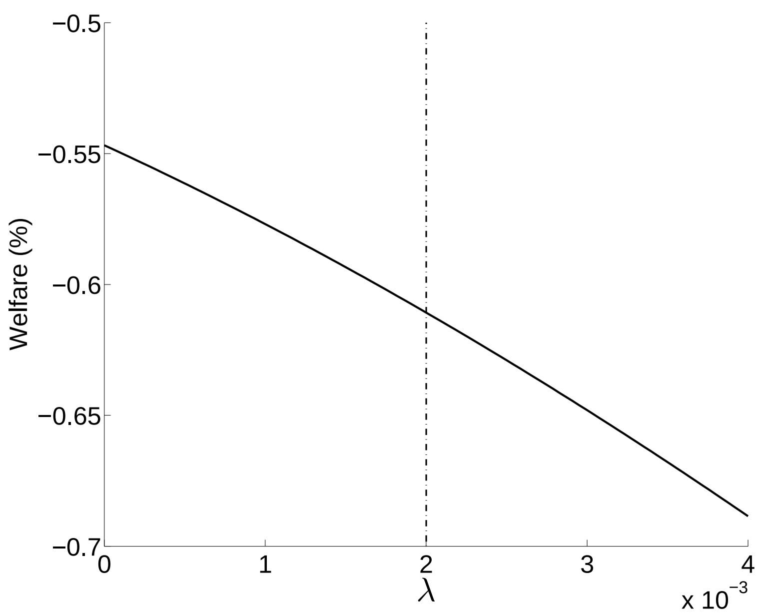
[14]While the optimal $\lambda$ is always zero, the welfare gains from conservatism do depend on these parameters. See the analysis in Appendix B.

Figure 1: Output gap, inflation, and nominal interest rate




Note: The figure displays how the output gap, the inflation rate, and the nominal interest rate in both states vary with $\lambda$. The dashed vertical lines indicate society's weight, $\bar{\lambda}$.

Figure 2: Welfare




Note: The figure displays how welfare varies with $\lambda$ in the two-state shock model. The dashed vertical lines indicate society's weight, $\bar{\lambda}$.

Consistent with Proposition 2, output and inflation in the high state are positive and negative, respectively, for any $\lambda$. The nominal interest rate is below the natural rate of interest, which is 4 percent. In the low state, output and inflation are negative, and the nominal interest rate is zero. Consistent with Proposition 3, as $\lambda$ decreases (i.e., as the central bank becomes more conservative), the deflation bias in the high state is reduced. This comes at the cost of a higher positive output gap in the high state, but a smaller deflation bias in the high state mitigates the decline in inflation and output in the low state.

The benefits of the smaller rate of deflation in the high state and larger output and inflation in the low state dominate the negative effect of a larger output gap in the high state. Accordingly, welfare increases with the degree of conservatism, as shown in Figure 2. Consistent with Proposition 4, the optimal weight is zero. In this case, the welfare gain is about 0.05 percent of the efficient level of consumption. Given that the welfare costs of business cycle fluctuations tend to be very small in this class of representative agent models, this number is non-trivial.[15] Further analysis in Appendix B shows that the size of the welfare gain increases with the frequency, persistence, and size of the crisis shock.

# 4 Extensions

In this section, we show that the desirability of inflation conservatism is robust to two model extensions. In the first extension, we augment the baseline discrete-state model with an additional shock that affects the supply side of the economy via the New Keynesian Phillips curve. In the second extension, we consider a continuous-state variant of the baseline model.

## 4.1 A model with demand and supply shocks

Thus far, the analysis has focused on an economy in which demand shocks are the only source of uncertainty. We now extend the analysis to an economy that is subject to both demand and supply shocks. In this case, the New Keynesian Phillips curve becomes

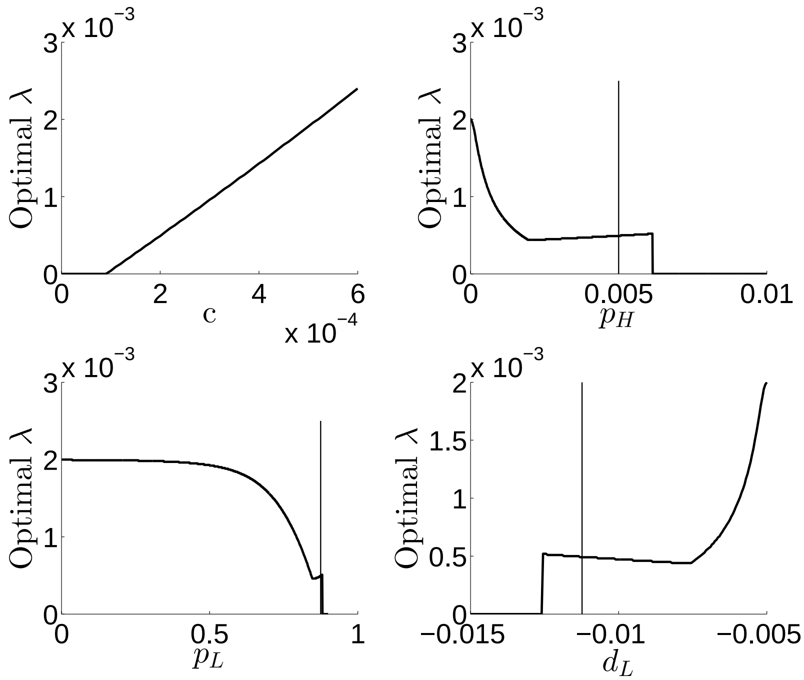
$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t, \tag{21}$$

where $u_t$ is a cost-push shock. We assume that the cost-push shock takes two values, $u_H = c \geq 0$ and $u_L = -c$, with probability 0.5 regardless of the state today.

The top-left panel of Figure 3 shows how the optimal weight varies with the size of the cost-push shock. While the optimal weight remains zero when the size of the cost-push shock is small, it is positive for a sufficiently large shock size and increases with the size of the shock.

[15]Notice that the welfare gains from eliminating the deflationary bias in the model with ZLB are not directly comparable to the benefits of eliminating the classical inflation bias discussed in the traditional monetary conservatism literature. The reason is that the classical inflation bias is the result of a deterministic steady state distortion whereas the deflation bias results from an only occasionally binding constraint.

Figure 3: Optimal weight in the model with cost-push shocks




Note: The figure displays how the optimal weight on output stabilization $\lambda$ depends on the size of the cost-push shock, and the frequency, persistence and size of the demand shock. Solid vertical lines indicate the baseline calibration.

There are two reasons why the optimal weight increases with the size of the cost-push shock. First, the optimal weight is equal to the social weight in the model with i.i.d. cost-push shocks but without demand shocks or the ZLB (see, for example, Clarida, Gali, and Gertler (1999)), while it is zero in the model without cost-push shocks but with demand shocks and the ZLB. Thus, a key determinant of the optimal weight in the model with both demand and cost-push shocks is the relative importance of these two shocks: the larger the size of the cost-push shock, the closer the optimal weight is to the social weight. Second, a lower weight on the output gap implies larger fluctuations in the nominal interest rate in states in which the demand shock is positive, $d_t = d_H$. If the size of the cost-push shock is sufficiently large, then for small weights on output stabilization $\lambda$ the ZLB can bind in the high-demand shock, deflationary cost-push shock state as well. In such a case, reducing $\lambda$ further does not increase the inflation rate in the high-demand shock, deflationary cost-push shock state, and at the same time detrimentally lowers the inflation rate in the high-demand shock, inflationary cost-push shock state, thereby leading to a larger deflation bias. In the particular parameterization shown in the figure, the optimal weight increases with the size of the cost-push shock for the second reason.

One consequence of this result is that the optimal weight decreases with the frequency, persistence, and severity of the demand shock. This can be seen in the top-right, bottom-left, and bottom-right panels of Figure 3, which show how the optimal weight varies with the frequency, persistence, and severity of the demand shock when the size of the cost-push shock is held at $\frac{0.02}{100}$.

The optimal weight coincides with the social weight when these parameters are zero, and it declines as these parameters, and thus the relative importance of the demand shock, increase. Interestingly, the effects of these parameters on the optimal weight are not monotonic. This is because reducing the weight on output stabilization can reduce welfare when doing so leads to excessive fluctuations in the nominal interest rate and causes the ZLB to bind in one of the high-demand shock states. For sufficiently small values of $p_H$, $p_L$, and $|d_L|$, the optimal weight can be so large that the ZLB does not bind anywhere near the optimal weight. Marginal increases in these parameters mean more severe deflation bias and optimal weights are lower as a result. For intermediate values of $p_H$, $p_L$, and $|d_L|$, marginal increases in these parameters do not lead to a reduction in the optimal weight because a reduction in the weight placed on output stabilization causes the ZLB to bind in one of the high-demand shock states. When $p_H$, $p_L$, and $|d_L|$ are sufficiently large, the benefit of reducing the deflation bias dominates the adverse consequences of hitting the ZLB in one of the high-demand shock states, and the optimal weight on output stabilization becomes zero. Appendix C extends the analysis to a model with persistent cost-push shocks.

### 4.2 A continuous-state model

We next examine whether the results from the analysis of the baseline model with a two-state Markov process for the demand shock also hold true when the shock is allowed to assume a continuum of values. Specifically, we assume that $d_t$ follows a stationary AR(1)-process

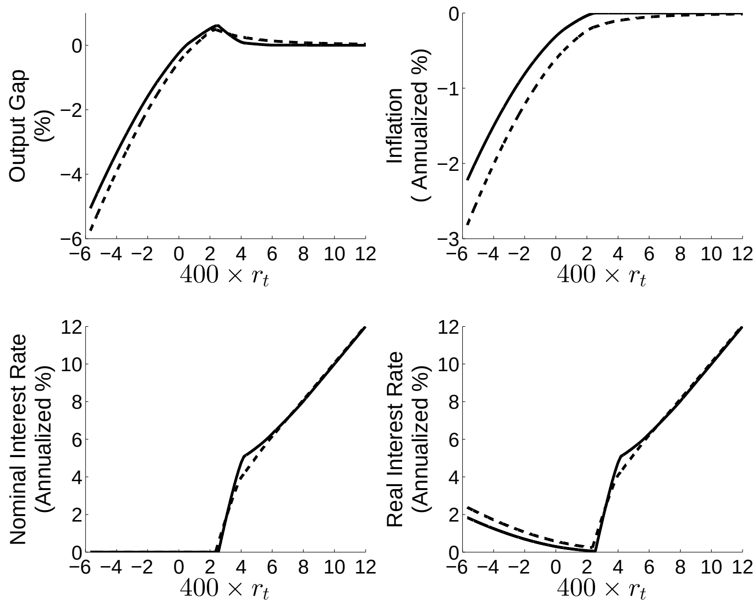
$$d_t = \rho_d d_{t-1} + \epsilon_t^d, \tag{22}$$

where the parameter $\rho_d$ represents the persistence of the shock and $\epsilon_t^d$ is an i.i.d. $\mathrm{N}\left(0, \sigma_\epsilon^2\right)$ innovation. We employ a projection method to approximate the policy functions numerically. One key reason for solving the model using the projection method is that this method does not assume certainty equivalence. The accurate treatment of expectation terms is crucial for our analysis. The details of the solution algorithm are described in Appendix D. As in the two-state shock model, structural parameters are set according to Eggertsson and Woodford (2003). We set the persistence parameter to 0.9. The standard deviation of the natural rate shock is set so that the probability of being at the ZLB is 30 percent, which is broadly in line with the U.S. experience over the past two decades.

Figure 4 shows the approximated policy functions for two alternative central bank regimes. Under the first regime, the central bank focuses solely on inflation stabilization (i.e., the conservative regime) as was shown to be optimal in the two-state model. Under the second regime, the central bank's preferences are identical to those of society as a whole, that is, $\lambda = \bar{\lambda} = 0.002$ (i.e., the benchmark regime).

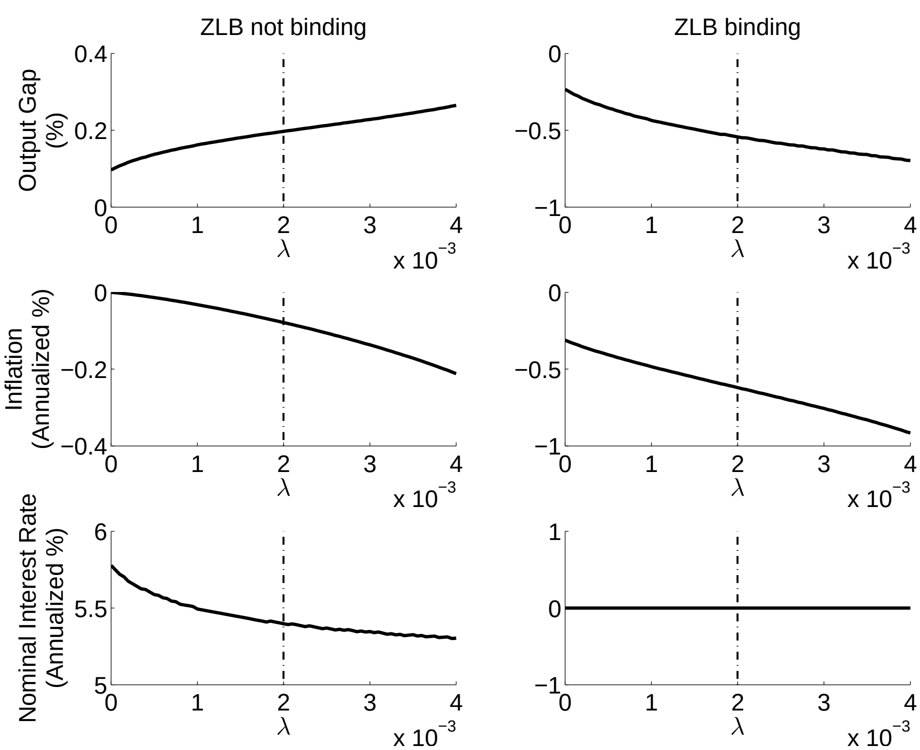
When the economy is in a liquidity trap and the natural real rate is negative, inflation and the output gap in both the conservative regime and the benchmark regime are negative, but as in the two-state model, the decline in the two target variables is less severe if the central bank is headed

Figure 4: Approximated policy functions (Continuous-state model)




Note: The figure displays the approximated policy functions for $\lambda = 0$ (solid lines) and $\lambda = \bar{\lambda}$ (dashed lines).

Figure 5: Conditional expectations (Continuous-state model)

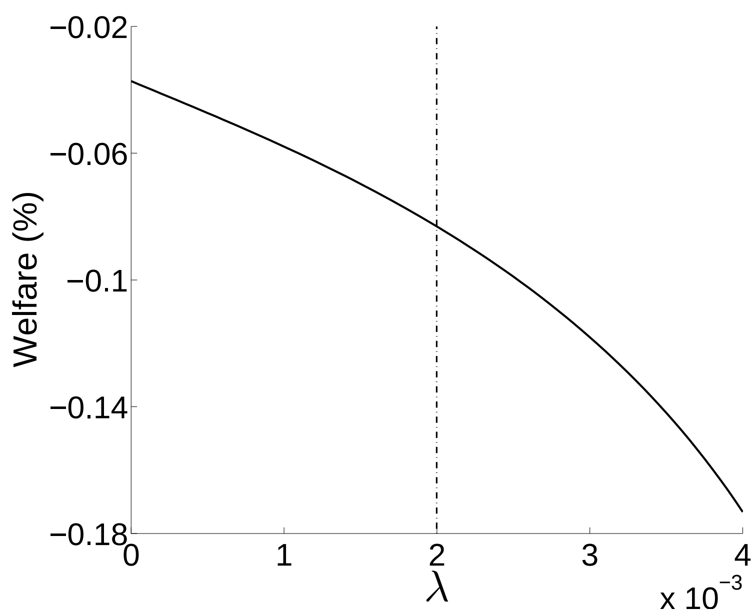



Note: The figure displays how the conditional averages of the output gap, the inflation rate and the nominal interest rate vary with $\lambda$. The dashed-dotted line indicates $\bar{\lambda}$.

by a conservative policymaker. The larger the adverse shock, the more pronounced is the difference in the equilibrium responses between the two regimes. Note, however, that in the continuous-state model, the ZLB is binding not only when the natural real rate of interest is negative but also when it is close to zero and positive.[16] In these states, inflation remains below zero but the output gap can be either negative or positive. The central bank can offset the direct effect of the natural rate shock but runs into the ZLB when trying to counteract the combined impact of the natural rate shock and the downward bias in agents' expectations. Finally, away from the ZLB, the conservative central banker perfectly stabilizes inflation at zero. Since inflation expectations are negative in all states of the world, inflation stability requires a positive output gap. In contrast, under the benchmark regime the economy is plagued by deflationary bias (i.e., negative inflation rates).

Figure 5 shows how the average inflation rate, output gap and nominal interest rate vary with the central banker's weight on output gap stabilization in those states in which the ZLB is not binding (left column) and in which the ZLB is binding (right column). The dashed vertical lines represent the averages associated with $\lambda = \bar{\lambda}$. The results are very similar to those in the two-state model; however, unlike in the numerical example for the two-state model, the average output gap away from the ZLB is reduced as $\lambda$ decreases, reflecting a strong feedback mechanism between policy actions and stabilization outcomes away from the ZLB and stabilization outcomes at the ZLB.

Figure 6: Welfare (Continuous-state model)




Note: The figure displays how welfare as defined in (12) varies with $\lambda$. The dashed-dotted line indicates $\bar{\lambda}$.

Finally, Figure 6 shows how welfare as defined in (12) depends on the central banker's preference

[16] In the two-state model this case was ruled out by the assumption that the nominal interest rate is strictly positive when the economy is in the normal state.

parameter $\lambda$. The benchmark regime with $\lambda = \bar{\lambda}$ is indicated by the dashed-dotted line. The welfare results from the baseline model continue to hold. First, the presence of the occasionally binding ZLB makes it desirable for society to appoint a conservative central banker. Second, the best-performing central banker puts zero weight on output gap stability.

# 5 Discussion

Throughout the paper, we have focused on analyzing how appointing a conservative central banker can mitigate the deflationary bias. An alternative way to reduce the deflationary bias is to appoint a central banker who has a higher inflation target. In an unreported exercise, we confirm that imposing a slightly higher inflation target does improve welfare. However, we refrain from a thorough discussion of the benefits of raising the inflation target because it has been extensively studied in other papers, albeit primarily in the context of simple interest-rate feedback rules.

We do want to note that the deflationary bias is a factor that affects the cost-benefit calculation of changing the inflation target. In the continuous-state framework, an increase in the inflation target will reduce the likelihood of hitting the ZLB and therefore reduce the deflationary bias. Some of the existing papers that work with perfect-foresight models are unable to capture this particular benefit of raising the inflation target. Analyzing how the deflationary bias affects the optimal inflation target would be an interesting venue for future research.

We also want to point out that, while both monetary conservatism and a higher inflation target can reduce the welfare costs associated with the ZLB constraint, they lack an important feature of optimal commitment policy—history-dependence. History-dependence allows policymakers to directly steer private sector expectations. Such expectations management is a particularly powerful tool to cope with ZLB events, as emphasized by Krugman (1998), Eggertsson and Woodford (2003) and Adam and Billi (2006). One problem with history-dependent policies is that they are not time consistent in general. Absent an explicit commitment technology or reputational force, discretionary central bankers are unable to credibly render monetary policy after a liquidity trap contingent on the state of the economy during the trap. To what extent alternative delegation schemes can generate the type of history-dependence seen in optimal commitment policy is a subject of our ongoing research Nakata and Schmidt (2015).

# 6 Conclusion

We have demonstrated, both analytically and numerically, that an economy that experiences occasional ZLB episodes can improve welfare by appointing a conservative central banker who is more concerned with inflation stabilization relative to output stabilization than society is. In the absence of policy commitment, optimal monetary policy suffers from a deflationary bias. Inflation stays below target even when the policy rate is positive because households and firms anticipate that the ZLB can be binding in the future. Subdued inflation rates away from the ZLB in turn exacerbate the decline in output and inflation when the economy is in a liquidity trap. A conservative central banker counteracts this vicious cycle by mitigating the deflationary bias away from the ZLB, thereby improving stabilization outcomes at and away from the ZLB.

As a byproduct of our analysis, we provide a closed-form characterization of the conditions that guarantee the existence of the standard as well as the deflationary Markov-Perfect equilibrium for the discrete-state version of our model.

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# Appendix

## A Proofs

In this section, we will provide details of the proofs in the main text. Since the proofs are algebraically intensive, we will have to omit some details in this section.

### A.1 Proof of Proposition 1

The standard Markov-Perfect equilibrium is given by a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the following system of linear equations—

$$y_H = \big[(1-p_H)y_H + p_H y_L\big] + \sigma\big[(1-p_H)\pi_H + p_H\pi_L - i_H + r^*\big] + d_H \tag{A.1}$$

$$\pi_H = \kappa y_H + \beta\big[(1-p_H)\pi_H + p_H\pi_L\big] \tag{A.2}$$

$$0 = \lambda y_H + \kappa\pi_H \tag{A.3}$$

$$y_L = \big[(1-p_L)y_H + p_L y_L\big] + \sigma\big[(1-p_L)\pi_H + p_L\pi_L - i_L + r^*\big] + d_L \tag{A.4}$$

$$\pi_L = \kappa y_L + \beta\big[(1-p_L)\pi_H + p_L\pi_L\big] \tag{A.5}$$

and

$$i_L = 0 \tag{A.6}$$

—and satisfies the following two inequality constraints:

$$i_H > 0 \tag{A.7}$$

and

$$\phi_L < 0. \tag{A.8}$$

$\phi_L$ denotes the Lagrangean multiplier on the ZLB constraint in the low state:

$$\phi_L := \lambda y_L + \kappa\pi_L. \tag{A.9}$$

We first prove four preliminary propositions (Propositions 1.A–1.D), then use them to prove the main proposition (Proposition 1) on the necessary and sufficient conditions for the existence of the standard Markov Perfect equilibrium.

Let

$$A(\lambda) := -\beta\lambda p_H, \tag{A.10}$$

$$B(\lambda) := \kappa^2 + \lambda(1-\beta(1-p_H)), \tag{A.11}$$

$$C := \frac{(1-p_L)}{\sigma\kappa}(1-\beta p_L + \beta p_H) - p_L, \tag{A.12}$$

$$D := -\frac{(1-p_L)}{\sigma\kappa}(1-\beta p_L + \beta p_H) - (1-p_L) = -1-C, \tag{A.13}$$

and

$$E(\lambda) := A(\lambda)D - B(\lambda)C. \tag{A.14}$$

**Assumption 1.A: $E(\lambda) \neq 0$.**

Throughout the proof, we will assume that Assumption 1.A holds.

**Proposition 1.A: There exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (A.1)–(A.6).**

*Proof*:

Rearranging the system of equations (A.1)–(A.6) and eliminating $y_H$ and $y_L$, we obtain two unknowns for $\pi_H$ and $\pi_L$ in two equations:

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ C & D \end{bmatrix} \begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} = \begin{bmatrix} 0 \\ r_L \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} = \frac{1}{A(\lambda)D - B(\lambda)C} \begin{bmatrix} D & -B(\lambda) \\ -C & A(\lambda) \end{bmatrix} \begin{bmatrix} 0 \\ r_L \end{bmatrix}, \tag{A.15}$$

where $r_L = r^* + \frac{1}{\sigma} d_L$.

Thus,

$$\pi_H := \frac{A(\lambda)}{E(\lambda)} r_L \tag{A.16}$$

and

$$\pi_L := \frac{-B(\lambda)}{E(\lambda)} r_L. \tag{A.17}$$

From the Phillips curves in both states, we obtain

$$y_H = \frac{\beta\kappa p_H}{E(\lambda)} r_L \tag{A.18}$$

and

$$y_L = -\frac{(1-\beta p_L)\kappa^2 + (1-\beta)(1+\beta p_H - \beta p_L)\lambda}{\kappa E(\lambda)} r_L. \tag{A.19}$$

**Proposition 1.B: Suppose (A.1)–(A.6) are satisfied. Then $\phi_L < 0$ if and only if $E(\lambda) < 0$.**

*Proof:* Notice that

$$\begin{aligned}\phi_L &= -\lambda\frac{(1-\beta p_L)\kappa^2 + (1-\beta)(1+\beta p_H - \beta p_L)\lambda}{\kappa E(\lambda)} r_L + \kappa\frac{-B(\lambda)}{E(\lambda)} r_L \\ &= -\left[\frac{\lambda}{\kappa}\big[(1-\beta p_L)\kappa^2 + (1-\beta)(1+\beta p_H - \beta p_L)\lambda\big] + \kappa B(\lambda)\right]\frac{r_L}{E(\lambda)}. \end{aligned} \tag{A.20}$$

Notice also that $r_L < 0$, $(1-\beta p_L)\kappa^2 > 0$, $(1-\beta)(1+\beta p_H - \beta p_L)\lambda \geq 0$, and $\kappa B(\lambda) > 0$. Thus, if $\phi_L < 0$, then $E(\lambda) < 0$. Similarly, if $E(\lambda) < 0$, then $\phi_L < 0$.

**Proposition 1.C: $E(\lambda) < 0$ if and only if $p_L^* < (\Theta_{-p_L})$.**

*Proof:* It is convenient to view $E(\cdot)$ as a function of $p_H$ and $p_L$ instead of $\lambda$ for a moment.

$$\begin{aligned} E(p_H, p_L) &= \beta\lambda p_H - \Gamma\left[\frac{1-p_L}{\sigma\kappa}(1-\beta p_L + \beta p_H) - p_L\right] \\ &= \beta\lambda p_H - \Gamma\left[\frac{1}{\sigma\kappa}(1-\beta p_L + \beta p_H - p_L + \beta p_L^2 - \beta p_H p_L) - p_L\right] \\ &= -\Gamma\frac{1}{\sigma\kappa}\beta p_L^2 + \Gamma\left[\frac{1}{\sigma\kappa}(1+\beta+\beta p_H) + 1\right]p_L + \beta\lambda p_H - \Gamma\frac{1}{\sigma\kappa}(1+\beta p_H) \\ &:= q_2 p_L^2 + q_1 p_L + q_0, \end{aligned} \tag{A.21}$$

where $\Gamma := \kappa^2 + \lambda(1-\beta)$ and

$$q_0 := \beta\lambda p_H - \Gamma\frac{1}{\sigma\kappa}(1+\beta p_H), \tag{A.22}$$

$$q_1 := \Gamma\big[\frac{1}{\sigma\kappa}(1+\beta+\beta p_H) + 1\big] > 0, \tag{A.23}$$

and

$$q_2 := -\Gamma\frac{1}{\sigma\kappa}\beta < 0. \tag{A.24}$$

This function, $E(\cdot,\cdot)$, has the following properties.

**Property 1: $E(p_H, 1) > 0$ for any $0 \leq p_H \leq 1$.**

*Proof:*

$$
\begin{aligned}
E(p_H, 1) &= -\Gamma \frac{1}{\sigma\kappa}\beta + \Gamma \left[ \frac{1}{\sigma\kappa}(1 + \beta + \beta p_H) + 1 \right] + \beta\lambda p_H - \Gamma \frac{1}{\sigma\kappa}(1 + \beta p_H) \\
&= \Gamma + \beta\lambda p_H > 0 \qquad \text{(A.25)}
\end{aligned}
$$

**Property 2:** $E(p_H, p_L)$ **is maximized at** $p_L > 1$ **for any** $0 \leq p_H \leq 1$**.**

*Proof:*

$$
\begin{aligned}
\frac{\partial E(p_H, p_L)}{\partial p_L} &= 2q_2 p_L^* + q_1 = 0 \\
\Leftrightarrow p_L^* &= -\frac{q_1}{2q_2} \\
&= \frac{\Gamma \left[ \frac{1}{\sigma\kappa}(1 + \beta + \beta p_H) + 1 \right]}{2\Gamma \frac{1}{\sigma\kappa}\beta} \\
&= \frac{\left[ \frac{1}{\sigma\kappa}(2\beta + (1 - \beta) + \beta p_H) + 1 \right]}{2\frac{1}{\sigma\kappa}\beta} > 1. \qquad \text{(A.26)}
\end{aligned}
$$

These two properties imply i) one root of $E(\cdot, p_L)$ is below 1 and ii) $E(\cdot, p_L) < 0$ below this root. Let's call this root $p_L^*(\Theta_{-p_L})$. $p_L^*(\Theta_{-p_L})$ is given by

$$
p_L^*(\Theta_{-p_L}) := \frac{-q_1 + \sqrt{q_1^2 - 4q_2 q_0}}{2q_2}. \qquad \text{(A.27)}
$$

If $E(\lambda) < 0$, then $p_L < p_L^*(\Theta_{-p_L})$. Similarly, if $p_L < p_L^*(\Theta_{-p_L})$, then $E(\lambda) < 0$. This completes the proof of Proposition 1.C. Note that Proposition 1.C holds independently of whether the system of linear equations (A.1)–(A.6) is satisfied or not.

**Proposition 1.D: Suppose (A.1)–(A.6) are satisfied and** $E(\lambda) < 0$**. Then** $i_H > 0$ **if and only if** $p_H < p_H^*(\Theta_{-p_H})$**.**

*Proof:*

First, notice that $i_H$ is given by

$$
\begin{aligned}
i_H =& r_H + \frac{1}{\sigma}\left[-p_H y_H + p_H y_L\right] + \left[(1-p_H)\pi_H + p_H \pi_L\right] \\
=& r_H + \frac{1}{\sigma} p_H \frac{-(1-\beta p_L)\kappa - (1-\beta)(1+\beta p_H - \beta p_L)\lambda/\kappa - \beta\kappa p_H}{E(\lambda)} r_L \\
&+ (1-p_H)\frac{A(\lambda)}{E(\lambda)} r_L + p_H \frac{-B(\lambda)}{E(\lambda)} r_L \\
=& -\frac{r_L}{E(\lambda)}\frac{\beta\Gamma}{\sigma\kappa} p_H^2 - \frac{r_L}{E(\lambda)}\left[\frac{(1-\beta p_L)\Gamma}{\sigma\kappa} + \kappa^2 + \lambda\right] p_H + r_H.
\end{aligned} \tag{A.28}
$$

Since $E(\lambda) < 0$, $i_H > 0$ requires

$$
\begin{aligned}
& r_L \frac{\beta\Gamma}{\sigma\kappa} p_H^2 + r_L \left(\frac{(1-\beta p_L)\Gamma}{\sigma\kappa} + \kappa^2 + \lambda\right) p_H - r_H E(\lambda) > 0 \\
\Leftrightarrow\ & r_L \frac{\beta\Gamma}{\sigma\kappa} p_H^2 + \left[r_L \left(\frac{(1-\beta p_L)\Gamma}{\sigma\kappa} + \kappa^2 + \lambda\right) - r_H \beta\lambda + r_H \Gamma\beta \frac{1-p_L}{\sigma\kappa}\right] p_H \\
& + r_H \Gamma \left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right) > 0.
\end{aligned} \tag{A.29}
$$

Dividing both sides by $\Gamma$ and by $-r_L$, we obtain

$$
\begin{aligned}
& -\frac{\beta}{\sigma\kappa} p_H^2 - \left[\frac{(1-\beta p_L) + (1-p_L)\beta\frac{r_H}{r_L}}{\sigma\kappa} + \frac{\kappa^2 + (1-\beta\frac{r_H}{r_L})\lambda}{\Gamma}\right] p_H \\
& - \left[\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right]\frac{r_H}{r_L} > 0.
\end{aligned} \tag{A.30}
$$

Let

$$
P(p_H) := \phi_2 p_H^2 + \phi_1 p_H + \phi_0, \tag{A.31}
$$

where

$$
\phi_0 := -\left[\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right]\frac{r_H}{r_L} \tag{A.32}
$$

$$
\phi_1 := -\frac{(1-\beta p_L) + (1-p_L)\beta\frac{r_H}{r_L}}{\sigma\kappa} - \frac{\kappa^2 + (1-\beta\frac{r_H}{r_L})\lambda}{\Gamma} \tag{A.33}
$$

and

$$
\phi_2 := -\frac{\beta}{\sigma\kappa} < 0. \tag{A.34}
$$

**Property 1:** $\phi_0 > 0$

*Proof:* Notice that $i_H = r_H > 0$ when $p_H = 0$. Since $E(\lambda) < 0$, the sign of $i_H$ is the same as the sign of $\phi_2 p_H^2 + \phi_1 p_H + \phi_0$. Thus, $\phi_0 > 0$. This completes the proof of Property 1.

$\phi_0 > 0$ and $\phi_2 < 0$ imply that one root of (A.31) is non-negative and $i_H > 0$ if and only if $p_H$ is below this non-negative root, given by

$$p_H^*(\Theta_{-p_H}) := \frac{-\phi_1 - \sqrt{\phi_1^2 - 4\phi_0\phi_2}}{2\phi_2}. \tag{A.35}$$

This completes the proof of Proposition 1.D.

With these four preliminary propositions (1.A–1.D), we are ready to prove our Proposition 1.

**Proposition 1: There exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the system of linear equations (A.1)–(A.6) and satisfies $\phi_L < 0$ and $i_H > 0$ if and only if $p_L < p_L^*(\Theta_{-p_L})$ and $p_H < p_H^*(\Theta_{-p_H})$.**

*Proof of "if" part:* Suppose that $p_L < p_L^*(\Theta_{-p_L})$ and $p_H < p_H^*(\Theta_{-p_H})$. According to Proposition 1.A there exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (A.1)–(A.6). According to Propositions 1.B and 1.C, $E(\lambda) < 0$ and $\phi_L < 0$. According to Proposition 1.D and the fact that $E(\lambda) < 0$, $i_H > 0$. This completes the proof of the "if" part.

*Proof of "only if" part:* Suppose that $\phi_L < 0$ and $i_H > 0$. According to Proposition 1.A there exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (A.1)–(A.6). According to Propositions 1.B and 1.C, $E(\lambda) < 0$ and $p_L < p_L^*(\Theta_{-p_L})$. According to Proposition 1.D and the fact that $E(\lambda) < 0$, $p_H < p_H^*(\Theta_{-p_H})$. This completes the proof of the "only if" part.

## A.2 Proof of Proposition 2

Proposition 2 characterizes the sign of inflation and output in both states. Using i) the restriction on $E(\lambda)$ (i.e. $E(\lambda) < 0$), ii) $r_L < 0$, and iii) inequalities on $A(\lambda)$, $B(\lambda)$, $C$, and $D$ given by equations (A.10)–(A.13), it is straightforward to check that:

$$\pi_H = \frac{A(\lambda)}{E(\lambda)} r_L \leq 0, \tag{A.36}$$

$$\pi_L = \frac{-B(\lambda)}{E(\lambda)} r_L < 0, \tag{A.37}$$

$$y_H = \frac{\beta \kappa p_H}{E(\lambda)} r_L > 0, \tag{A.38}$$

and

$$y_L = -\frac{(1-\beta p_L)\kappa^2 + (1-\beta)(1+\beta p_H - \beta p_L)\lambda}{\kappa E(\lambda)} r_L < 0. \tag{A.39}$$

Inflation and output are negative in the low state. Inflation in the high state is negative, which is what we call deflation bias. Positive output in the high state is consistent with negative inflation in the high state and the optimality condition of the central bank (i.e., equation (15)).

## A.3 Proof of Proposition 3

Proposition 3 characterizes how $\lambda$ affects inflation and output in both states.

$$\begin{aligned}
\frac{\partial \pi_H}{\partial \lambda} &:= \frac{A'(\lambda)E(\lambda) - A(\lambda)E'(\lambda)}{E(\lambda)^2} r_L \\
&= \frac{A(\lambda)B'(\lambda) - A'(\lambda)B(\lambda)}{E(\lambda)^2} C r_L \\
&= \frac{-\beta p_H \lambda \left(1-\beta+\beta p_H\right) + \beta p_H \left(\kappa^2 + \left(1-\beta+\beta p_H\right)\lambda\right)}{E(\lambda)^2} C r_L \\
&= \frac{\beta p_H \kappa^2}{E(\lambda)^2} C r_L < 0,
\end{aligned} \tag{A.40}$$

where $A'(\lambda)$ and $B'(\lambda)$ denote the partial derivatives of $A(\cdot)$ and $B(\cdot)$ with respect to $\lambda$.

$$\begin{aligned}
\frac{\partial \pi_L}{\partial \lambda} &:= \frac{-B'(\lambda)E(\lambda) + B(\lambda)E'(\lambda)}{E(\lambda)^2} r_L \\
&= \frac{A'(\lambda)B(\lambda) - A(\lambda)B'(\lambda)}{E(\lambda)^2} D r_L \\
&= -\frac{\beta p_H \kappa^2}{E(\lambda)^2} D r_L < 0
\end{aligned} \tag{A.41}$$

$$\begin{aligned}
\frac{\partial y_H}{\partial \lambda} &:= \frac{-\beta\kappa p_H E'(\lambda)}{E(\lambda)^2} r_L \\
&= -\frac{\beta\kappa p_H \left(A'(\lambda)D - B'(\lambda)C\right)}{E(\lambda)^2} r_L \\
&= -\frac{\beta\kappa p_H}{E(\lambda)^2} \left[\beta p_H - (1-\beta)C\right] r_L \\
&= -\frac{\beta\kappa p_H}{E(\lambda)^2} \left[\beta p_H - (1-\beta)\left(\frac{1-p_L}{\kappa\sigma}\left(1-\beta p_L+\beta p_H\right) - p_L\right)\right] r_L
\end{aligned} \tag{A.42}$$

$$\begin{aligned}
\frac{\partial y_L}{\partial \lambda} &:= -\frac{(1-\beta)(1-\beta p_L+\beta p_H)E(\lambda) - \big((1-\beta p_L)\kappa^2 + (1-\beta)(1-\beta p_L+\beta p_H)\lambda\big)\, E'(\lambda)}{\kappa E(\lambda)^2} r_L \\
&= -\Bigg[\frac{(1-\beta)(1-\beta p_L+\beta p_H)(A(\lambda)D - B(\lambda)C)}{\kappa E(\lambda)^2} \\
&\quad - \frac{\big((1-\beta p_L)\kappa^2 + (1-\beta)(1-\beta p_L+\beta p_H)\lambda\big)\left(A'(\lambda)D - B'(\lambda)C\right)}{\kappa E(\lambda)^2}\Bigg] r_L \\
&= \frac{\beta\kappa p_H}{E(\lambda)^2}\left[(1-\beta)C + (1-\beta p_L)\right] r_L < 0
\end{aligned} \tag{A.43}$$

## A.4 Proof of Proposition 4

Proposition 4 states that welfare is maximized at $\lambda = 0$.

Society's unconditional expected value is given by

$$EV(\lambda) = \frac{1}{1-\beta}\left[\frac{1-p_L}{1-p_L+p_H} u(\pi_H, y_H) + \frac{p_H}{1-p_L+p_H} u(\pi_L, y_L)\right]. \tag{A.44}$$

To show that $E[V(\lambda)]$ is maximized at $\lambda = 0$, we show that

$$\frac{\partial EV(\lambda)}{\partial \lambda} < 0 \tag{A.45}$$

for all $\lambda \geq 0$.

The derivative of the unconditional expected value is given by

$$\frac{\partial EV}{\partial \lambda} = \frac{1}{1-\beta}\left[\frac{1-p_L}{1-p_L+p_H}\frac{\partial u(\pi_H, y_H)}{\partial \lambda} + \frac{p_H}{1-p_L+p_H}\frac{\partial u(\pi_L, y_L)}{\partial \lambda}\right]. \tag{A.46}$$

The partial derivatives of society's utility are given by

$$
\begin{aligned}
\frac{\partial u(\pi_H, y_H)}{\partial \lambda} &:= \frac{\partial}{\partial \lambda}\left[-\frac{1}{2}(\bar{\lambda} y_H(\lambda)^2 + \pi_H(\lambda)^2)\right] \\
&= -\left(\bar{\lambda}\frac{\beta\kappa p_H}{E(\lambda)} r_L \left(-\frac{\beta\kappa p_H E'(\lambda)}{E(\lambda)^2} r_L\right) + \frac{A(\lambda)}{E(\lambda)} r_L \frac{A'(\lambda)E(\lambda) - A(\lambda)E'(\lambda)}{E(\lambda)^2} r_L\right) \\
&= -\frac{\bar{\lambda}\beta^2\kappa^2 p_H^2 E'(\lambda) + \lambda\beta^2 p_H^2 E(\lambda) - \lambda^2\beta^2 p_H^2 E'(\lambda)}{E(\lambda)^3} r_L^2 \\
&= \frac{\beta^2\kappa^2 p_H^2}{E(\lambda)^3}\left(\bar{\lambda}E'(\lambda) + \lambda C\right) r_L^2 \\
&= \frac{\beta^2\kappa^2 p_H^2}{E(\lambda)^3}\left[\bar{\lambda}\left(\beta p_H - (1-\beta)C\right) + \lambda C\right] r_L^2. \qquad \text{(A.47)}
\end{aligned}
$$

Note that we have already shown that the sign of the first term in square brackets, $\beta p_H - (1-\beta)C$, determines the sign of $\frac{\partial y_H}{\partial \lambda}$. If $\beta p_H - (1-\beta)C > 0$, then $\frac{\partial y_H}{\partial \lambda} > 0$ and $\frac{\partial u(\pi_H, y_H)}{\partial \lambda} > 0$. If instead $\beta p_H - (1-\beta)C < 0$, then the sign of $\frac{\partial u(\pi_H, y_H)}{\partial \lambda}$ is ambiguous.

$$
\begin{aligned}
\frac{\partial u(\pi_L, y_L)}{\partial \lambda} &:= \frac{\partial}{\partial \lambda}\left[-\frac{1}{2}(\bar{\lambda} y_L(\lambda)^2 + \pi_L(\lambda)^2)\right] \\
&= \frac{\beta p_H}{E(\lambda)^3}\Big(\bar{\lambda}\left[(1-\beta p_L)\kappa^2 + (1-\beta)(1+\beta p_H - \beta p_L)\lambda\right]\left[(1-\beta)C + (1-\beta p_L)\right] \\
&+ \kappa^2\left[\kappa^2 + \lambda(1-\beta(1-p_H))\right](1+C)\Big) r_L^2 \\
&:= \frac{\beta p_H}{E(\lambda)^3}\left(\bar{\lambda}\Phi_1(\lambda) + \Phi_2(\lambda)\right) r_L^2 < 0, \qquad \text{(A.48)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_1(\lambda) &:= \Phi_{1,1} + \Phi_{1,2}\lambda, \qquad \text{(A.49)} \\
\Phi_2(\lambda) &:= \Phi_{2,1} + \Phi_{2,2}\lambda, \qquad \text{(A.50)}
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_{1,1} &:= (1-\beta p_L)\kappa^2\left[(1-\beta)C + (1-\beta p_L)\right] > 0, \qquad \text{(A.51)} \\
\Phi_{1,2} &:= \left[(1-\beta)C + (1-\beta p_L)\right](1-\beta)(1+\beta p_H - \beta p_L) > 0, \qquad \text{(A.52)} \\
\Phi_{2,1} &:= \kappa^4(1+C) > 0, \qquad \text{(A.53)}
\end{aligned}
$$

and

$$
\Phi_{2,2} := \kappa^2(1-\beta(1-p_H))(1+C) > 0. \qquad \text{(A.54)}
$$

Hence,

$$
\begin{aligned}
\frac{\partial EV}{\partial \lambda} &= \frac{(1-\beta)^{-1} r_L^2}{1-p_L+p_H}\left((1-p_L)\frac{\beta^2\kappa^2 p_H^2}{E(\lambda)^3}\left[\bar{\lambda}\left(\beta p_H-(1-\beta)C\right)+\lambda C\right]+p_H\frac{\beta p_H}{E(\lambda)^3}\left(\bar{\lambda}\Phi_1(\lambda)+\Phi_2(\lambda)\right)\right) \\
&= \frac{(1-\beta)^{-1}\beta p_H^2 r_L^2}{(1-p_L+p_H)E(\lambda)^3}\left(\beta\kappa^2(1-p_L)\left[\bar{\lambda}\left(\beta p_H-(1-\beta)C\right)+\lambda C\right]+\left(\bar{\lambda}\Phi_1(\lambda)+\Phi_2(\lambda)\right)\right). \quad \text{(A.55)}
\end{aligned}
$$

Let

$$
\Omega(\lambda) := \beta\kappa^2(1-p_L)(\beta p_H-(1-\beta)C)\bar{\lambda}+\beta\kappa^2(1-p_L)C\lambda+\bar{\lambda}(\Phi_{1,1}+\Phi_{1,2}\lambda)+\Phi_{2,1}+\Phi_{2,2}\lambda. \quad \text{(A.56)}
$$

If $\Omega(\lambda) > 0$ for all $\lambda \geq 0$, then $\frac{\partial EV(\lambda)}{\partial \lambda} < 0$ for all $\lambda \geq 0$. Notice that $\Omega'(\lambda)$ is positive since the coefficients on $\lambda$ are all positive. Thus, we only need to show $\Omega(0) > 0$ to show that $\Omega(\lambda) > 0$ for all $\lambda \geq 0$.

$$
\begin{aligned}
\Omega(0) &= \beta\kappa^2(1-p_L)\left[\beta p_H-(1-\beta)C\right]\bar{\lambda}+\bar{\lambda}\Phi_{1,1}+\Phi_{2,1} \\
&= \left[\beta\kappa^2(1-p_L)\left[\beta p_H-(1-\beta)C\right]+\Phi_{1,1}\right]\bar{\lambda}+\Phi_{2,1} \\
&= \left[\beta^2\kappa^2(1-p_L)p_H+(1-\beta)^2\kappa^2 C+(1-\beta p_L)^2\kappa^2\right]\bar{\lambda}+\kappa^4(1+C) > 0, \quad \text{(A.57)}
\end{aligned}
$$

given that $C > 0$ for the equilibrium to exist and $\bar{\lambda} > 0$. This completes the proof.

# B Welfare gains of conservatism

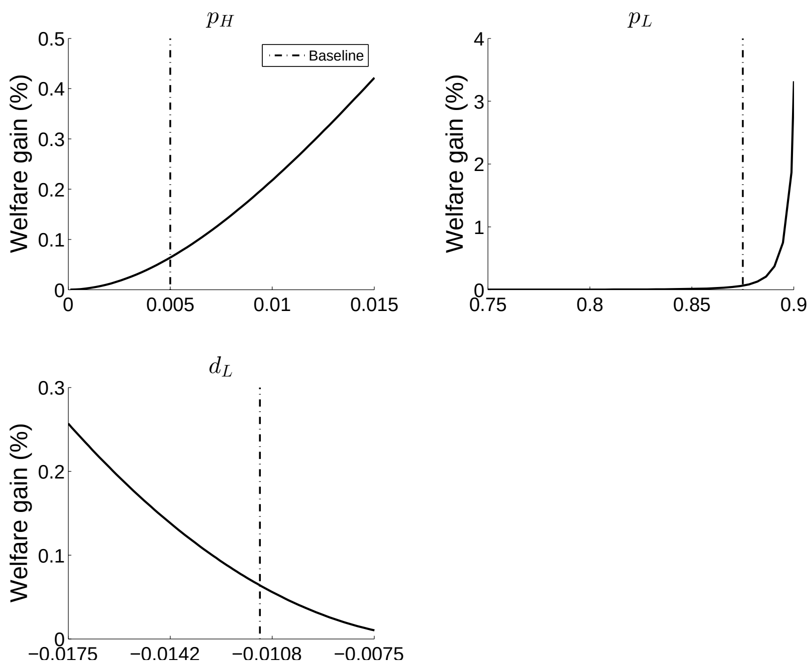
In the baseline calibration of the two-state shock model, the welfare gain of appointing a fully conservative central banker is about 0.05 percent of the efficient level of consumption. This number clearly depends on the frequency, persistence, and severity of the ZLB episodes. Figure 7 shows this dependency. The top-left panel shows that the welfare gain of conservatism increases with the frequency of the shock, and it reaches about 0.3 percent when the frequency is 1.5 percent. According to the top-right panel, the welfare gain increases sharply with persistence, exceeding 2 percent at $p_L = 0.9$. Finally, the welfare gain increases with the severity of the shock (the absolute value of $d_L$), as shown in the bottom-left panel.

# C The model with persistent cost-push shocks

In the main text, we analyzed how the introduction of i.i.d. cost-push shocks affects the optimal weight placed on the output gap stability term. In this section, we relax the i.i.d. assumption to consider persistent cost-push shocks. In particular, we consider cases in which the probability of staying at the high (or low) cost-push state tomorrow when today's cost-push state is high (or low) is either 0.6 or 0.8, as opposed to 0.5 in the baseline i.i.d. case.

In the top-left panel of Figure 8, the blue and red dashed lines show how the optimal weight on

Figure 7: Welfare gains of conservatism (Two-state shock model)

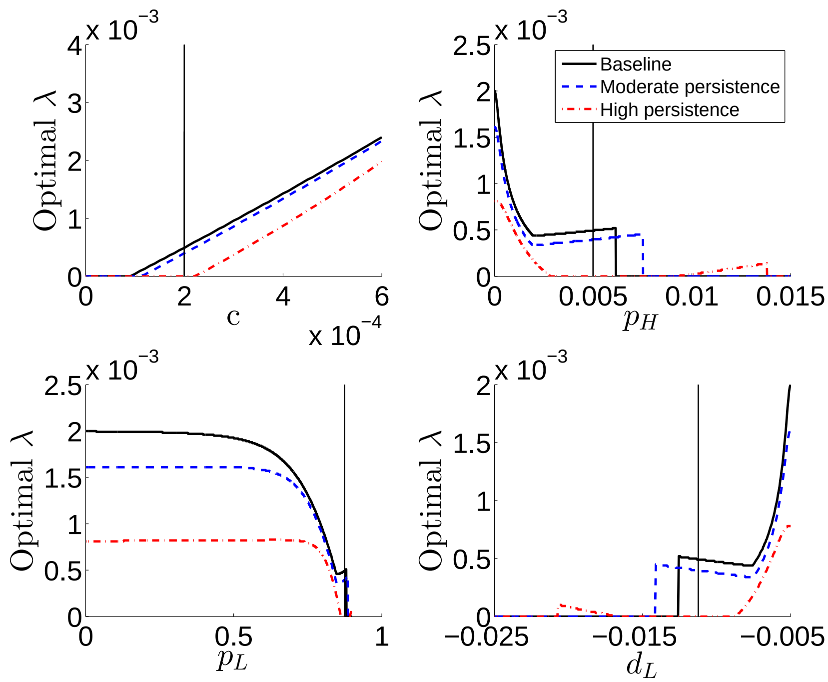



Note: The figure displays how the welfare gains of conservatism ($\lambda = 0$) relative to the baseline regime ($\lambda = \bar{\lambda}$) depend on the frequency, persistence and size of the demand shock.

the output gap volatility term varies with the size of the cost-push shocks in the economies with persistence of 0.6 and 0.8, respectively. The black line is for the optimal weight in the baseline economy with non-persistent cost-push shocks. The panel shows that the optimal weight is lower when cost-push shocks are more persistent. The reason is as follows. In the model *without* the ZLB, the optimal weight is the same as the true weight if the shock is not persistent. When cost-push shocks are persistent, the optimal weight is smaller than the true weight because inflation is expected to be non-zero in the future and a smaller weight on the output gap stabilization term reduces the deviation of expected inflation from zero. As such, the more persistent the cost-push shocks are, the smaller the optimal weight is. As we saw in the main text, the introduction of the ZLB makes it desirable to place a lower weight on the output gap stabilization term. The fact that the optimal weight is lower when cost-push shocks are more persistent in the model with the ZLB is inherited from the same feature in the model without the ZLB.

The top-right and bottom two panels shows how the optimal weight varies with the frequency, persistence, and size of the crisis shock. Consistent with the baseline model with non-persistent cost-push shocks, the optimal weight declines as the frequency, persistence, and severity of the crisis increases with the same non-monotonicity discussed in the main text. Consistent with our discussion in the previous paragraph, the optimal weight is generally lower when cost-push shocks are more persistent. However, due to the non-monotonicity, there are regions of parameter values under which the optimal weight is larger with more persistent cost-push shocks. For example, the

Figure 8: Optimal weight in the model with persistent cost-push shocks




Note: The figure displays how the optimal weight on output stabilization $\lambda$ depends on the size of the cost-push shock, and the frequency, persistence and size of the demand shock. Under the baseline case, the probability of staying at the high (low) cost-push state tomorrow when today's cost-push state is high (low) is 0.5. Under the "moderate" and "high" persistence cases, that probability is 0.6 and 0.8., respectively.

optimal weight is larger with persistence of 0.8 than with persistence of 0.6 for the values of $p_H$ between 0.01 and 0.013 and for the values of $d_L$ between -0.021 and -0.018.

# D Computational algorithm for the continuous-state model

Let $Z = [\pi, y]'$ and $\widetilde{Z} = [Z', i]'$. We approximate $Z$ by a linear combination of $n$ basis functions $\psi_i$, $i = 1, ..., n$. In matrix notation,

$$Z(d) \approx C\Psi(d), \tag{D.1}$$

where

$$C = \begin{pmatrix} c_1^\pi & \cdots & c_n^\pi \\ c_1^y & \cdots & c_n^y \end{pmatrix}, \qquad \Psi(d) = \begin{pmatrix} \psi_1(d) \\ \vdots \\ \psi_n(d) \end{pmatrix}.$$

The coefficients $c_i^j$, $i = 1, 2, ..., n$; $j \in \{\pi, y\}$, are set such that (D.1) holds exactly at $n$ selected collocation nodes collected in vector $\overline{d}$

$$Z\left(\overline{d}_{(k)}\right) = C\Psi\left(\overline{d}_{(k)}\right),$$

for $k = 1, ..., n$, where $\overline{d}_{(k)}$ refers to the $k$th element of $\overline{d}$. We use linear splines as basis functions and choose the breakpoints such that they coincide with the collocation nodes.

The iterative solution algorithm to obtain the policy function approximations then works as follows. We start with an initial guess on the coefficient matrix $C^{(0)}$. For fixed $C^{(s)}$ in iteration $s$, we first update the expectations functions,

$$E\pi^{(s)}\left(\overline{d}_{(k)}\right) = \sum_{l=1}^{m} \varpi_l C_{(1,:)}^{(s)} \Psi\left(\rho_d \overline{d}_{(k)} + \epsilon_{(l)}\right)$$

and

$$Ey^{(s)}\left(\overline{d}_{(k)}\right) = \sum_{l=1}^{m} \varpi_l C_{(2,:)}^{(s)} \Psi\left(\rho_d \overline{d}_{(k)} + \epsilon_{(l)}\right),$$

for $k = 1, ..., n$. We use a Gaussian quadrature scheme to discretize the normally distributed random variable, where $\epsilon$ is a $m \times 1$ matrix of quadrature nodes and $\varpi$ is a vector of length $m$ containing the weights.

Assuming first that the zero bound is not binding at any collocation node, the optimality conditions for the discretionary policy regime imply

$$\tilde{Z}^{(s)}\left(\overline{d}_{(k)}\right) = A^{-1} \cdot B + A^{-1} \cdot F \cdot EZ^{(s)}\left(\overline{d}_{(k)}\right) + A^{-1} \cdot D \cdot \overline{d}_{(k)},$$

for $k = 1, ..., n$, where

$$A = \begin{pmatrix} 1 & -\kappa & 0 \\ 0 & 1 & \sigma \\ \kappa & \lambda & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \sigma r^* \\ 0 \end{pmatrix}, \qquad F = \begin{pmatrix} \beta & 0 \\ \sigma & 1 \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For those $k$ for which the zero lower bound is violated—i.e., $i^{(s)}\left(\overline{d}_{(k)}\right) < 0$—matrix $A$ in the update is replaced by

$$\hat{A} = \begin{pmatrix} 1 & -\kappa & 0 \\ 0 & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix}.$$

We then update $C^{(s+1)} = \left[Z^{(s)}\left(\overline{d}_{(1)}\right) \quad \cdots \quad Z^{(s)}\left(\overline{d}_{(n)}\right)\right]$ and continue the iteration procedure until $\left\|vec\left(C^{(s+1)} - C^{(s)}\right)\right\|_\infty < \delta$.
The collocation nodes are equally distributed with a support covering $\pm$ 4 unconditional standard deviations of the exogenous state variable. We use MATLAB routines from the CompEcon toolbox of Miranda and Fackler (2002) to obtain the Gaussian quadrature approximation of the innovations to the demand shock, and to evaluate the spline functions.

# E Existence of other Markov-Perfect Equilibria

While we focus on the standard Markov-Perfect equilibrium in which the ZLB constraint binds in the low state but not in the high state, there are potentially three other types of Markov-Perfect equilibria: i) one in which the ZLB constraint binds in both states (*the deflationary Markov-Perfect equilibrium*), ii) one in which the ZLB constraint does not bind in both states (*the ZLB-free Markov-Perfect equilibrium*), and iii) one in which the ZLB binds in the high state but not in the low state (*the topsy-turvy Markov-Perfect equilibrium*). In this section, we examine whether and under what conditions any of these other types of Markov-Perfert equilibria exist. Our main results are that i) the conditions for the existence of the deflationary Markov-Perfect equilibrium are the same as those for the existence of the standard Markov-Perfect equilibrium and ii) the other two types do not exist under any parameter configurations.[17]

[17] There is a continuum of sunspot equilibria which may randomly move between the standard and deflationary Markov-Perfect equilibria. Characterizing the conditions for the existence of such sunspot equilibria is outside the scope of the paper.

## E.1 Existence of the deflationary Markov-Perfect equilibrium

The deflationary Markov-Perfect equilibrium is given by a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the following system of linear equations—

$$y_H = \big[(1-p_H)y_H + p_H y_L\big] + \sigma\big[(1-p_H)\pi_H + p_H\pi_L - i_H + r^*\big] + d_H, \tag{E.1}$$

$$\pi_H = \kappa y_H + \beta\big[(1-p_H)\pi_H + p_H\pi_L\big], \tag{E.2}$$

$$i_H = 0, \tag{E.3}$$

$$y_L = \big[(1-p_L)y_H + p_L y_L\big] + \sigma\big[(1-p_L)\pi_H + p_L\pi_L - i_L + r^*\big] + d_L, \tag{E.4}$$

$$\pi_L = \kappa y_L + \beta\big[(1-p_L)\pi_H + p_L\pi_L\big], \tag{E.5}$$

and

$$i_L = 0, \tag{E.6}$$

—and satisfies the following two inequality constraints:

$$\phi_H < 0 \tag{E.7}$$

and

$$\phi_L < 0. \tag{E.8}$$

$\phi_H$ and $\phi_L$ denote the Lagrangean multipliers on the ZLB constraint in the high state and in the low state:

$$\phi_H := \lambda y_H + \kappa\pi_H \tag{E.9}$$

and

$$\phi_L := \lambda y_L + \kappa\pi_L. \tag{E.10}$$

The following proposition states that the conditions for the existence of the deflationary Markov-Perfect equilibrium are identical to the conditions for the existence of the standard Markov-Perfect equilibrium.

**Proposition 5**: The deflationary Markov-Perfect equilibrium exists if and only if

$$p_L \leq p_L^*(\Theta_{(-p_L)})$$

and

$$p_H \leq p_H^*(\Theta_{(-p_H)}),$$

where the cutoff values $p_L^*(\Theta_{(-p_L)})$ and $p_H^*(\Theta_{(-p_H)})$ are defined by (A.27) and (A.35) in Appendix A. We first prove six preliminary propositions, then use them to prove Proposition 5.

Let

$$\tilde{A} := -\left(\frac{p_H}{\sigma\kappa}\left(1 - \beta p_L + \beta p_H\right) + p_H\right), \tag{E.11}$$

$$\tilde{B} := -\tilde{A} - 1, \tag{E.12}$$

and

$$\begin{aligned}\tilde{E} &:= \tilde{A}D - \tilde{B}C \\ &= -\tilde{A} + C,\end{aligned} \tag{E.13}$$

where $C$ and $D < 0$ are defined in (A.12) and (A.13).

**Assumption 5.A: $\tilde{E} \neq 0$.**

Throughout the proof, we will assume that Assumption 5.A holds.

**Proposition 5.A: There exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (E.1)–(E.6).**

*Proof*:

Rearranging the system of equations (E.1)–(E.6) and eliminating $y_H$ and $y_L$, we obtain two unknowns for $\pi_H$ and $\pi_L$ in two equations:

$$\begin{aligned}\begin{bmatrix} \tilde{A} & \tilde{B} \\ C & D \end{bmatrix}\begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} &= \begin{bmatrix} r_H \\ r_L \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} &= \frac{1}{\tilde{E}}\begin{bmatrix} D & -\tilde{B} \\ -C & \tilde{A} \end{bmatrix}\begin{bmatrix} r_H \\ r_L \end{bmatrix},\end{aligned} \tag{E.14}$$

where $r_L = r^* + \frac{1}{\sigma}d_L$ and $r_H = r^* + \frac{1}{\sigma}d_H$.

Thus,

$$\pi_H := \frac{\tilde{A}}{\tilde{E}}r_L - \frac{C}{\tilde{E}}r_H \tag{E.15}$$

and

$$\pi_L := \frac{-\tilde{B}}{\tilde{E}}r_L + \frac{D}{\tilde{E}}r_H. \tag{E.16}$$

From the Phillips curves in both states, we obtain

$$y_H = \frac{(1-\beta)\tilde{A} - \beta p_H}{\kappa\tilde{E}}r_L - \frac{(1-\beta)C - \beta p_H}{\kappa\tilde{E}}r_H \tag{E.17}$$

and

$$y_L = \frac{(1-\beta)\tilde{A} + (1-\beta p_L)}{\kappa \tilde{E}} r_L - \frac{(1-\beta)C + (1-\beta p_L)}{\kappa \tilde{E}} r_H. \tag{E.18}$$

**Proposition 5.B: Suppose (E.1)–(E.6) are satisfied. Then $\phi_L < 0$ only if $\tilde{E} > 0$.**

*Proof by contradiction*:

First, notice that

$$\phi_L = \frac{1}{\tilde{E}} \left[ -(1+C)\kappa r_H - (1-\beta)C\frac{\lambda}{\kappa} r_H - (1-\beta p_L)\frac{\lambda}{\kappa} r_H + (1+\tilde{A})\kappa r_L + \frac{\lambda}{\kappa}(1-\beta)\tilde{A} r_L + \frac{\lambda}{\kappa}(1-\beta p_L) r_L \right]. \tag{E.19}$$

Suppose that $\tilde{E} < 0$. From the equation above we know that, given $\tilde{E} < 0$, $\phi_L < 0$ if and only if

$$-(1+C)\kappa r_H - (1-\beta)C\frac{\lambda}{\kappa} r_H - (1-\beta p_L)\frac{\lambda}{\kappa} r_H + (1+\tilde{A})\kappa r_L + \frac{\lambda}{\kappa}(1-\beta)\tilde{A} r_L + \frac{\lambda}{\kappa}(1-\beta p_L) r_L > 0. \tag{E.20}$$

Collecting terms, this condition can be simplified to

$$\left( \kappa + \frac{\lambda}{\kappa}(1-\beta p_L) \right) [(1+A) r_L - (1+C) r_H] > 0. \tag{E.21}$$

From (E.13), we know that $\tilde{E} < 0$ if and only if $C < \tilde{A}$, where $\tilde{A} < 0$. Furthermore, from (A.12) we know that $C > -1$. Suppose $C \to -$; then $A > -1$, which proves that (E.21) cannot hold.

**Proposition 5.C: Suppose (E.1)-(E.6) are satisfied and $\tilde{E} > 0$. Then $\phi_L < 0$ if $\phi_H < 0$.**

*Proof*: This follows directly from noticing that

$$\phi_L = \phi_H + \frac{\kappa^2 + \lambda}{\kappa \tilde{E}} (r_L - r_H). \tag{E.22}$$

**Proposition 5.D: Suppose (E.1)–(E.6) are satisfied and $\tilde{E} > 0$. Then $\phi_H < 0$ if and only if $p_H < p_H^*(\Theta_{-p_H})$.**

*Proof*:

First, notice that

$$\phi_H = \frac{1}{\tilde{E}}\left([-C\kappa - (1-\beta)\frac{\lambda}{\kappa}C + \beta\frac{\lambda}{\kappa}p_H]r_H + [\kappa\tilde{A} + (1-\beta)\frac{\lambda}{\kappa}\tilde{A} - \beta\frac{\lambda}{\kappa}p_H]r_L\right). \tag{E.23}$$

Since $\tilde{E} > 0$, $\phi_H < 0$ requires

$$-C\kappa - (1-\beta)\frac{\lambda}{\kappa}C + \beta\frac{\lambda}{\kappa}p_H]r_H + [\kappa\tilde{A} + (1-\beta)\frac{\lambda}{\kappa}\tilde{A} - \beta\frac{\lambda}{\kappa}p_H]r_L < 0. \tag{E.24}$$

Multiplying both sides by $\frac{\kappa}{\Gamma}\frac{1}{r_L}$ and collecting terms, we get

$$\begin{aligned}
&-\frac{\beta}{\sigma\kappa}p_H^2 - \left(\frac{(1-\beta p_L) + (1-p_L)\beta\frac{r_H}{r_L}}{\sigma\kappa} + \frac{\kappa^2 + (1-\beta\frac{r_H}{r_L})\lambda}{\Gamma}\right)p_H \\
&-\left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right)\frac{r_H}{r_L} > 0.
\end{aligned} \tag{E.25}$$

Let

$$P(p_H) := \phi_2 p_H^2 + \phi_1 p_H + \phi_0 : \tag{E.26}$$

where

$$\begin{aligned}
\phi_0 &:= -\left[\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right]\frac{r_H}{r_L} \\
\phi_1 &:= -\frac{(1-\beta p_L) + (1-p_L)\beta\frac{r_H}{r_L}}{\sigma\kappa} - \frac{\kappa^2 + (1-\beta\frac{r_H}{r_L})\lambda}{\Gamma} \\
\phi_2 &:= -\frac{\beta}{\sigma\kappa} < 0,
\end{aligned} \tag{E.27}$$

which is similar to the definition in Appendix A. $\phi_0 > 0$ and $\phi_2 < 0$ imply that one root of (E.26) is non-negative and $\phi_H < 0$ if and only if $p_H$ is below this non-negative root, given by

$$p_H^*(\Theta_{-p_H}) := \frac{-\phi_1 - \sqrt{\phi_1^2 - 4\phi_0\phi_2}}{2\phi_2}. \tag{E.28}$$

This completes the proof of Proposition 5.D.

**Proposition 5.E: $\tilde{E} > 0$ and $p_H < p_H^*(\Theta_{-p_H})$ only if $E(\lambda) < 0$.**

*Proof*:

Suppose that $\tilde{E} > 0$ and $p_H < p_H^*(\Theta_{-p_H})$. Then $\tilde{E} + P(p_H) > 0$.

$$
\begin{aligned}
\tilde{E} + P(p_H) = &\frac{\beta}{\sigma\kappa}p_H^2 + \left(1 + \frac{(1-\beta p_L) + (1-p_L)\beta}{\sigma\kappa}\right)p_H + \left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right) \\
&- \frac{\beta}{\sigma\kappa}p_H^2 - \left(\frac{(1-\beta p_L) + (1-p_L)\beta\frac{r_H}{r_L}}{\sigma\kappa} + \frac{\kappa^2 + (1-\beta\frac{r_H}{r_L})\lambda}{\Gamma}\right)p_H \\
&- \left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right)\frac{r_H}{r_L} \\
&= \left[\beta\frac{\lambda}{\Gamma}p_H - \beta\frac{1-p_L}{\sigma\kappa}p_H - \left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right)\right]\left(\frac{r_H}{r_L} - 1\right). \qquad \text{(E.29)}
\end{aligned}
$$

Since $\left(\frac{r_H}{r_L} - 1\right) < 0$, the following condition has to hold:

$$
\beta\frac{\lambda}{\Gamma}p_H - \beta\frac{1-p_L}{\sigma\kappa}p_H - \left(\frac{1-p_L}{\sigma\kappa}(1-\beta p_L) - p_L\right) < 0. \qquad \text{(E.30)}
$$

Collecting terms, we get

$$
-\Gamma\frac{1}{\sigma\kappa}\beta p_L^2 + \Gamma\left[\frac{1}{\sigma\kappa}(1+\beta+\beta p_H) + 1\right]p_L + \beta\lambda p_H - \Gamma\frac{1}{\sigma\kappa}(1+\beta p_H) = E(\lambda) < 0. \qquad \text{(E.31)}
$$

This completes the proof of Proposition 5.E. Note that Proposition 5.E holds independently of whether the system of linear equations (E.1)–(E.6) is satisfied or not.

**Proposition 5.F:** $E(\lambda) < 0$ **only if** $\tilde{E} > 0$**.**

*Proof*: This follows directly from noticing that

$$
\tilde{E} = -\frac{E(\lambda)}{\Gamma} + \frac{\beta\lambda}{\Gamma}p_H + \frac{1}{\sigma\kappa}\left(\beta p_H + 1 + \beta(1-p_L)\right)p_H. \qquad \text{(E.32)}
$$

Note that Proposition 5.F holds independently of whether the system of linear equations (E.1)–(E.6) is satisfied or not.

With these six preliminary propositions (5.A–5.F), we are ready to prove Proposition 5.

**Proposition 5: There exists a vector** $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ **that solves the system of linear equations (E.1)–(E.6) and satisfies** $\phi_L < 0$ **and** $\phi_H < 0$ **if and only if** $p_L < p_L^*(\Theta_{-p_L})$ **and** $p_H < p_H^*(\Theta_{-p_H})$**.**

*Proof of "if" part:* According to Proposition 5.A, there exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (E.1)–(E.6). Suppose that $p_L < p_L^*(\Theta_{-p_L})$ and $p_H < p_H^*(\Theta_{-p_H})$. According to Proposition 1.C (which does not rely on the system of linear equations), $E(\lambda) < 0$. According to Proposition

5.F, then $\tilde{E} > 0$. According to Proposition 5.D, this implies $\phi_H < 0$. Finally, according to Proposition 5.C, this implies $\phi_L < 0$. This completes the proof of "if" part.

*Proof of "only if" part:* According to Proposition 5.A, there exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (E.1)–(E.6). Suppose that $\phi_L < 0$ and $\phi_H < 0$. According to Proposition 5.B, $\tilde{E} > 0$. According to Proposition 5.D, then $p_H < p_H^*(\Theta_{-p_H})$. According to Proposition 5.E, this implies $E(\lambda) < 0$. According to Proposition 1.C (which does not rely on the system of linear equations), $p_L < p_L^*(\Theta_{-p_L})$. This completes the proof of the "only if" part.

## E.2 Nonexistence of the topsy-turvy Markov-Perfect equilibrium

The topsy-turvy Markov-Perfect equilibrium is given by a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the following system of linear equations—

$$y_H = \big[(1-p_H)y_H + p_H y_L\big] + \sigma\big[(1-p_H)\pi_H + p_H\pi_L - i_H + r^*\big] + d_H, \tag{E.33}$$
$$\pi_H = \kappa y_H + \beta\big[(1-p_H)\pi_H + p_H\pi_L\big], \tag{E.34}$$
$$i_H = 0, \tag{E.35}$$
$$y_L = \big[(1-p_L)y_H + p_L y_L\big] + \sigma\big[(1-p_L)\pi_H + p_L\pi_L - i_L + r^*\big] + d_L, \tag{E.36}$$
$$\pi_L = \kappa y_L + \beta\big[(1-p_L)\pi_H + p_L\pi_L\big], \tag{E.37}$$

$$0 = \lambda y_L + \kappa \pi_L, \tag{E.38}$$

—and satisfies the following two inequality constraints:

$$\phi_H < 0 \tag{E.39}$$

and

$$i_L > 0 \tag{E.40}$$

$\phi_H$ denotes the Lagrangean multiplier on the ZLB constraint in the high state:

$$\phi_H := \lambda y_H + \kappa \pi_H. \tag{E.41}$$

**Proposition 6**: The topsy-turvy Markov-Perfect equilibrium does not exist.

We first prove three preliminary propositions, then use them to prove Proposition 6.

Let

$$\hat{C}(\lambda) := \kappa^2 + \lambda\left(1 - \beta p_L\right), \tag{E.42}$$

$$\hat{D}(\lambda) := -\beta\lambda\left(1 - p_L\right), \tag{E.43}$$

and

$$\hat{E}(\lambda) := \tilde{A}\hat{D}(\lambda) - \tilde{B}\hat{C}(\lambda), \tag{E.44}$$

where $\tilde{A}$ and $\tilde{B}$ are defined in (E.11) and (E.12).

**Assumption 6.A: $\hat{E}(\lambda) \neq 0$.**

Throughout the proof, we will assume that Assumption 6.A holds.

**Proposition 6.A: There exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (E.33)–(E.38).**

*Proof*:

Rearranging the system of equations (E.33)–(E.38) and eliminating $y_H$ and $y_L$, we obtain two unknowns for $\pi_H$ and $\pi_L$ in two equations:

$$\begin{aligned} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} &= \begin{bmatrix} r_H \\ 0 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \pi_L \\ \pi_H \end{bmatrix} &= \frac{1}{\hat{E}(\lambda)} \begin{bmatrix} \hat{D}(\lambda) & -\tilde{B} \\ -\hat{C}(\lambda) & \tilde{A} \end{bmatrix} \begin{bmatrix} r_H \\ 0 \end{bmatrix}, \end{aligned} \tag{E.45}$$

where $r_H = r^* + \frac{1}{\sigma} d_H$.

Thus,

$$\pi_H := -\frac{\hat{C}(\lambda)}{\hat{E}(\lambda)} r_H \tag{E.46}$$

and

$$\pi_L := \frac{\hat{D}(\lambda)}{\hat{E}(\lambda)} r_H. \tag{E.47}$$

From the Phillips Curves in both states, we obtain

$$y_H = -\frac{(1-\beta)\hat{C}(\lambda) + \beta p_H \Gamma}{\kappa \hat{E}(\lambda)} r_H \tag{E.48}$$

and

$$y_L = -\frac{(1-\beta p_L)\hat{D}(\lambda) + (1-p_L)\beta\hat{C}(\lambda)}{\kappa \hat{E}(\lambda)} r_H. \tag{E.49}$$

**Proposition 6.B: Suppose (E.33)–(E.38) are satisfied. Then $\phi_H < 0$ if and only if $\hat{E}(\lambda) > 0$.**

*Proof*:

Notice that

$$\phi_H = -\frac{1}{\kappa \hat{E}(\lambda)} \Big( (\kappa^2 + (1-\beta))\hat{C}(\lambda) + \beta p_H \Gamma \Big) r_H, \tag{E.50}$$

where $(\kappa^2 + (1-\beta))\hat{C}(\lambda) + \beta p_H \Gamma > 0$ and $r_H > 0$. Hence, $\phi_H < 0$ if and only if $\hat{E}(\lambda) > 0$.

**Proposition 6.C: Suppose (E.33)–(E.38) are satisfied. Then $i_L > 0$ only if $\hat{E}(\lambda) < 0$.**

*Proof*:

Notice that

$$\begin{aligned} i_L &= r_L - \frac{1}{\kappa \hat{E}(\lambda)} \Big[ -\kappa p_L \hat{D}(\lambda) + \kappa(1-p_L)\hat{C}(\lambda) + \frac{1}{\sigma}(1-p_L) \\ &\Big( (1-\beta)\hat{C}(\lambda) + \beta p_H \Gamma + (1-\beta p_L)\hat{D}(\lambda) + (1-p_L)\beta \hat{C}(\lambda) \Big) \Big] r_H, && \text{(E.51)} \\ &= r_L - \frac{1}{\kappa \hat{E}(\lambda)} \Big[ -\kappa p_L \hat{D}(\lambda) + \kappa(1-p_L)\hat{C}(\lambda) + \frac{1}{\sigma}(1-p_L) \\ &\Big( (1-\beta p_L)\hat{C}(\lambda) + \beta p_H \Gamma + (1-\beta p_L)\hat{D}(\lambda) \Big) \Big], && \text{(E.52)} \\ &= r_L - \frac{1}{\kappa \hat{E}(\lambda)} \Big[ -\kappa p_L \hat{D}(\lambda) + \kappa(1-p_L)\hat{C}(\lambda) + \frac{1}{\sigma}(1-p_L)\left(\beta p_H + (1-\beta p_L)\right)\Gamma \Big] r_H, && \text{(E.53)} \end{aligned}$$

where $-\kappa p_L \hat{D}(\lambda) + \kappa(1-p_L)\hat{C}(\lambda) + \frac{1}{\sigma}(1-p_L)\left(\beta p_H + (1-\beta p_L)\right)\Gamma > 0$, $r_H > 0$, and $r_L < 0$. Hence, $i_L > 0$ only if $\hat{E}(\lambda) < 0$.

With these three preliminary propositions (6.A-6.C), we are ready to prove Proposition 6.

**Proposition 6: There exists no vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the system of linear equations (E.33)–(E.38) and satisfies $i_L > 0$, $\phi_H < 0$.**

*Proof by contradiction:* According to Proposition 6.A, there exists a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves (E.33)–(E.38). Suppose that $\phi_H < 0$ and $i_L > 0$. According to Proposition 6.B, $\phi_H < 0$ implies $\hat{E}(\lambda) > 0$. According to Proposition 6.C, $i_L > 0$ implies $\hat{E}(\lambda) < 0$, which contradicts $(i_L > 0,\ \phi_H < 0)$.

### E.3 Nonexistence of the ZLB-free Markov-Perfect equilibrium

The ZLB-free Markov-Perfect equilibrium is given by a vector $\{y_H, \pi_H, i_H, y_L, \pi_L, i_L\}$ that solves the following system of linear equations—

$$y_H = \big[(1-p_H)y_H + p_H y_L\big] + \sigma\big[(1-p_H)\pi_H + p_H\pi_L - i_H + r^*\big] + d_H, \tag{E.54}$$
$$\pi_H = \kappa y_H + \beta\big[(1-p_H)\pi_H + p_H\pi_L\big], \tag{E.55}$$
$$0 = \lambda y_H + \kappa\pi_H, \tag{E.56}$$
$$y_L = \big[(1-p_L)y_H + p_L y_L\big] + \sigma\big[(1-p_L)\pi_H + p_L\pi_L - i_L + r^*\big] + d_L, \tag{E.57}$$
$$\pi_L = \kappa y_L + \beta\big[(1-p_L)\pi_H + p_L\pi_L\big], \tag{E.58}$$
$$\tag{E.59}$$

and

$$0 = \lambda y_L + \kappa\pi_L, \tag{E.60}$$

—and satisfies the following two inequality constraints:

$$i_H > 0 \tag{E.61}$$

and

$$i_L > 0. \tag{E.62}$$

**Proposition 7**: The ZLB-free Markov-Perfect equilibrium does not exist.

*Proof*:

Let

$$\hat{E} = \big[1 - \beta(1-p_H) + \frac{\kappa^2}{\lambda}\big](1 - \beta p_L + \frac{\kappa^2}{\lambda}) - \beta^2 p_H(1-p_L). \tag{E.63}$$

**Assumption 7.A: $\hat{E} \neq 0$.**

Throughout the proof, we will assume that Assumption 7.A holds.

Notice that $i_H$ and $i_L$ only appear in the consumption Euler equations. Thus, we can first find a vector of $\{y_H, \pi_H, y_L, \pi_L\}$ that satisfies the Phillips curves and the government's optimality condition in both states, then use the two consumption Euler equations to find $i_H$ and $i_L$. Rearranging the system of equations (E.55), (E.56), (E.58), and (E.60) and eliminating $y_H$ and $y_L$, we obtain two unknowns for $\pi_H$ and $\pi_L$ in two equations:

$$\pi_H = -\frac{\kappa^2}{\lambda}\pi_H + \beta\big[(1-p_H)\pi_H + p_H\pi_L\big] \tag{E.64}$$

and

$$\pi_L = -\frac{\kappa^2}{\lambda}\pi_L + \beta\big[(1-p_L)\pi_H + p_L\pi_L\big] \tag{E.65}$$

$$\Rightarrow \begin{bmatrix} 1-\beta(1-p_H)+\frac{\kappa^2}{\lambda} & -\beta p_H \\ -\beta(1-p_L) & 1-\beta p_L+\frac{\kappa^2}{\lambda} \end{bmatrix} \begin{bmatrix} \pi_H \\ \pi_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{E.66}$$

$$\Rightarrow \begin{bmatrix} \pi_H \\ \pi_L \end{bmatrix} = \frac{1}{\hat{E}} \begin{bmatrix} 1-\beta p_L+\frac{\kappa^2}{\lambda} & \beta(1-p_L) \\ \beta p_H & 1-\beta(1-p_H)+\frac{\kappa^2}{\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{E.67}$$

From the Phillips curves in both states, we obtain

$$y_H = 0 \tag{E.68}$$

and

$$y_L = 0. \tag{E.69}$$

From the consumption Euler equations in both states, we obtain

$$i_H = r^* + \frac{d_H}{\sigma} > 0 \tag{E.70}$$

and

$$i_L = r^* + \frac{d_L}{\sigma} < 0. \tag{E.71}$$

These two inequalities hold because we assume that $d_H > -\sigma r^*$ and $d_L < -\sigma r^*$ (see Section 2 in the main text). Thus, the inequality condition for the policy rate in the low state is violated. Accordingly, there is no vector that solves (E.54)–(E.60) and satsifies both $i_H > 0$ and $i_L > 0$.