JSPS Grant-in-Aid for Scientific Research (S)

Central Bank Communication Design

working paper series

No.005 (January 2019)

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Stability of Sunspot Equilibria under Adaptive Learning with Imperfect Information[1]

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This version: January 31, 2019

[1]The authors are grateful to George W. Evans, Wouter den Haan, Ricardo Reis and participants at the University of Oregon Macro Group, LSE Faculty Work in Progress Seminar, the Econometric Society, the European Economic Association, the Midwest Macroeconomics Workshop, and the Royal Economic Society for their helpful comments and suggestions. For financial support, the authors thank Ishii Memorial Research Promotion Foundation, KAKENHI (Nos. 20730139, 15K03372), Kansai University Researcher 2013 & 2016, Murata Science Foundation, Nomura Foundation for Academic Promotion, and Zengin Foundation for Studies on Economics and Finance.

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Abstract

This paper investigates whether sunspot equilibria are stable under agents' adaptive learning with imperfect information sets of exogenous variables. Each exogenous variable is observable for a part of agents and unobservable from others so that agents' forecasting models are heterogeneously misspecified. The paper finds that stability conditions of sunspot equilibria are relaxed or unchanged by imperfect information. In a basic New Keynesian model with highly imperfect information, sunspot equilibria are stable if and only if nominal interest rate rules violate the Taylor principle. This result is contrast to the literature in which sunspot equilibria are stable only if policy rules follow the principle, and is consistent with the observations during past business cycles fluctuations.

JEL classification: C62; D83; E52

Keywords: Sunspot equilibria; Stability; Adaptive learning; Private information; Heterogeneous beliefs; Taylor principle

1 Introduction

Sunspot-driven business cycle models are popular tools to account for the features of macroeconomic fluctuations that are not explained by fundamental shocks. US business cycles in the pre-Volcker period are considered to be driven by self-fulfilling expectations, so-called "sunspots" (see Benhabib and Farmer, 1994; Farmer and Guo, 1994). Those non-fundamental expectations are considered to stem from the Fed's passive stance to inflation (see Clarida, Gali, and Gertler, 2000; Lubik and Schorfheide, 2004). Even recently, global financial turmoils in the last decade had historic magnitudes that could not be explained by fundamental reasons, and hence it is analyzed in models with sunspot expectations (see Benhabib and Wang, 2013; Gertler and Kiyotaki, 2015).[1]

In conjunction with the increasing focus on business cycle models with sunspot equilibria, the empirical plausibility of those models has been tested in more plausible frameworks of expectations formation than rational expectations (REs). A stream of the literature examines the stability of sunspot equilibria under *adaptive learning*, where it is assumed that agents have no knowledge of the structure of the economy enough to form rational expectations so that they form their expectations by estimating econometric models with available data (Evans and Honkapohja, 2001). Woodford (1990) is the first study that found stable sunspot equilibria under adaptive learning. Evans and McGough (2005c) show that non-fundamental equilibria can be stable under serially correlated sunspot shocks. Other studies provide stability conditions imposed on structural or policy parameters in standard business cycle models.

On the other hand, there has been found the so-called *stability puzzle*: that is, those parameter conditions are not satisfied in calibrated business cycle models (see Evans and McGough, 2005a). There has been the conventional wisdom that sunspot equilibria are unstable in real business cycle (RBC) models. Duffy and Xiao (2007) show that stability conditions obtained in RBC models contradict the parameter restrictions that

[1] Non-fundamental fluctuations are also empirically observed in the Great Recession in the US of the late 2000s (see Farmer, 2012a,b); the housing bubble in the US of the late 2000s (Miao and Wang, 2012); the European debt crisis in the early 2010s (Bacchetta, Tille, and van Wincoop, 2012).

ensure the *positive* feedback of expectations in those models.[2] A similar puzzle is observed in New Keyensian (NK) models. Evans and McGough (2005b) find that stability conditions, if any, include the *active* monetary policy rule that follows so-called the *Taylor principle*, that is, to raise the nominal interest rate more-than-one-for-one in response to an increase in the inflation rate. However, this result contradicts not only the positive feedback restriction, but also the historical evidence that during the past business cycles that were suspected to be non-fundamental, central banks adopted the *passive* policy rule that does not respond to inflation aggressively.[3] In this sense, the empirical plausibility of sunspot equilibria has not been fully confirmed in the framework of adaptive learning.

This paper analytically investigates the stability of sunspot equilibria under adaptive learning in standard business cycle models. Following the manners of Marcet and Sargent (1989a) and Nakagawa (2015), the paper incorporates the private information of exogenous variables that make agents' information sets limited and heterogeneous, and examines whether the stability conditions of sunspot equilibria are affected by the imperfect information of exogenous variables. Next, the paper analyzes a basic NK model with imperfect information to examine whether the imperfect information helps obtain empirically plausible stability conditions imposed on monetary policy rules.

Imperfect information of exogenous variables has been little considered in the analysis of sunspot equilibria under adaptive learning, but the imperfect information is reasonable to think of as a factor to affect the dynamics of sunspot equilibria. The empirical evidence of imperfect information of fundamental shocks has been well provided (e.g., Mankiw, Reis, and Wolfers, 2003; Madeira and Zafar, 2015), and in the literature of rational expectations, it is well demonstrated that self-fulfilling fluctuations are driven by the private information of fundamental shocks in financial

[2]See also Branch and McGough (2004), Evans and McGough (2005a), Shea (2013, 2016), and Ji and Xiao (2017). Exceptionally, McGough, Meng, and Xue (2013) show a situation where sunspot equilibria are stable in RBC models.

[3]*Negative* feedback conditions are obtained by Evans and McGough (2005c), Shea (2013, 2016), and Berardi (2015). Non-fundamental flucutuations under passive interest rate rules were suspected for the US during the 1970s (see Clarida, Gali, and Gertler, 2000; Lubik and Schorfheide, 2004; Belaygorod and Dueker, 2009), the EU during the 1980s and 1990s (Hirose, 2013), and the China during 1990s and 2000s (Zheng and Guo, 2013).

markets (e.g., Jermann and Quadrini, 2012; Benhabib and Wang, 2013; Gertler and Kiyotaki, 2015; Benhabib, Dong, and Wang, 2018). In addition, imperfect information of exogenous variables is more reasonable in the framework of adaptive learning than rational expectations because such imperfect information is highly possible in an environment where agents have no knowledge of the structure of the economy.[4] Accordingly, Nakagawa (2015) investigates the stability of a *fundamental* equilibrium with the imperfect information introduced in the present paper and shows that the stability is improved by the imperfect information of exogenous variables. The same mechanism is expected to work in the dynamics of sunspot equilibria as well.

This paper finds that the existence of imperfect information provides *nonnegative* effect on the stability of sunspot equilibria. Specifically, the stability conditions of sunspot equilibria are relaxed or unchanged with the degree of information imperfection. As a result, sunspot equilibria can be stable in calibrated business cycle models. In calibrated NK models, if information is highly imperfect, sunspot equilibria are stable if and only if nominal interest rate rules violate the Taylor principle. This result resolves the stability puzzle in the sense that they are consistent with the observations during past business cycles fluctuations.

Our paper is closely related to the literature on sunspot *restricted perceptions equilibria* (RPEs), where agents' information sets of economic variables are limited.[5] The advantage of our model is to define the degrees of limitation and heterogeneity of agents' information sets so that describes a broad class of information structures, not only limited information sets, but also, for example, the private information that has been considered by Marcet and Sargent (1989a).[6] In addition, the paper pro-

[4]The difficulty in identifying the processes of fundamental shocks without such knowledge are well recognized in the time-series literature (e.g., Giannone and Reichlin, 2006; Alessi, Barigozzi, and Capasso, 2011). The difficulty might be demonstrated by the fact that the consensus about the source of the Great Depression in the 1930s was established more than six decades later (see Eichengreen, 1992).

[5]The related literature also considers the partial or asymmetric information of exogenous variables (e.g., Adam, 2003; Branch, McGough, and Zhu, 2017), the asymmetric information of endogenous variables (e.g., Adam, Evans, and Honkapohja, 2006), and the partial or asymmetric information of sunspot variables (e.g., Guse, 2005; Berardi, 2009).

[6]Marcet and Sargent (1989a) analyze the *perpetually and symmetrically uninformed* model in the framework of adaptive learning, and this model is originally developed

vides analytical results on the relationships between those degrees and the stability of sunspot equilibria. Thus, the results of our paper are applicable to models with a variety of imperfect information of exogenous variables.[7]

The contribution of the paper is to find a relationship between the imperfect information of fundamental shocks and self-fulfilling fluctuations in the framework of adaptive learning. In the RE framework, many studies demonstrate that sunspot equilibria stem from the information imperfection of fundamental shocks. In particular, after the financial turmoils in the last decade, self-fulfilling fluctuations are considered to be driven by the financial frictions that are induced by the existence of private information of fundamental shocks in financial markets. In their frameworks, the imperfect information makes sunspot equilibria stationary. In our framework, the imperfect information makes those equilibria stable.

Finally, our results reinforce the nonnegative effect of imperfect information on the stability of an equilibrium. In the same information framework, Nakagawa (2015) shows that the stability of a *fundamental* equilibrium is similarly affected by the information imperfection. Thus, the present paper suggests that regardless of a fundamental or sunspot equilibrium, the information imperfection of exogenous variables has nonnegative effects on its stability.

The paper is structured as follows. The next section presents our model and stationary sunspot equilibria attainable under rational expectations. Section 3 provides a benchmark analysis under adaptive learning with perfect information to confirm conventional results. Section 4 obtains the stability conditions under imperfect information and clarify the effect of imperfect information on the stability of sunspot equilibria. Section 5 provides stability conditions in a multivariate model. Section 6 applies our analytical results to a New Keynesian model. Finally, the

by Lucas (1972), Townsend (1983), and Pearlman and Sargent (2005). Note that the stability of sunspot equilibria was beyond the scope of the researches by Marcet and Sargent (1989a).

[7]Our framework is different from the *dynamic predictor selection* model, which premises homogeneity in agents' information sets such that all agents choose among the same list of econometric models (see Brock and Hommes, 1997; Branch and Evans, 2006; Berardi, 2015).

paper presents our conclusions.

2 Model

To clarify the stability of sunspot equilibria under imperfect information intuitively, this paper starts with a univariate reduced-form linear expectations model. The analysis using a multivariate model will be shown in Section 5.

The economy is represented by the dynamics of one endogenous variable y_t and the vector of serially correlated exogenous variables $w_t = (w_{1t}, \ldots, w_{nt})'$.

$$y_t = \beta E_t^* y_{t+1} + \gamma w_t, \tag{1}$$
$$w_t = \Phi w_{t-1} + v_t. \tag{2}$$

The standard deviation of w_{it} for each $i \in \{1, \cdots, n\}$ is defined by $\sigma_{ii} > 0$, and the correlation matrix of w_t is defined by $\Gamma \equiv \left(\rho_{ij}\right)_{1 \leq i,j \leq n}$, where $\rho_{ij} \in [0, 1]$ denotes the correlation between w_i and w_j, and $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1$ for each $i, j \in \{1, \ldots, n\}$. v_t is the $n \times 1$ vector of fundamental shocks with means of zero that drive the stochastic process of w_t.[8] The parameter β is a nonzero scholar coefficient of $E_t^* y_{t+1}$, $\gamma = (\gamma_1, ..., \gamma_n)$ is a $1 \times n$ coefficient matrix of w_t, and Φ is an $n \times n$ matrix of autocorrelation coefficients of w_t. E_t^* is the operator of the expectation of y_{t+1} at time t, which is not necessarily rational.

For ease of calculation, we impose regularity assumptions on these parameters (see Appendix A): in particular,

$$\Phi \equiv diag\left(\varphi_i\right)_{1 \leq i \leq n} \geq 0 \text{ and } \Gamma \geq 0,$$

where $0 \leq \varphi_i < 1$ for all i. These assumptions are not crucial for our analysis, as most stationary linear models in the literature can be transformed to satisfy these conditions.

To obtain stability conditions that are consistent with calibrated business cycle models, the *positive* feedback of expectations is assumed as a

[8]Note that exogenous variables with nonzero and heterogeneous means can be transformed to the form (1).

plausible parameter restriction:

$$\beta > 0. \tag{3}$$

Later in a multivariate model (Section 5), a corresponding restriction will be assumed. Section 6.4 will show that calibrated NK models have this feature.

In the framework of rational expectations ($E_t^* = E_t$), a solution to the system should be a stochastic process y_t that satisfies Eqs. (1)–(2) and has the *general form* (GF) representation:

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1} \gamma w_{t-1} + \varepsilon_t, \tag{4}$$

where $\varepsilon_{t+1} \equiv y_{t+1} - E_t y_{t+1}$ is a martingale difference sequence (mds) representing for agents' forecast error and $E_t \varepsilon_{t+1} = 0$. Under Eq. (3), if and only if

$$\beta > 1, \tag{5}$$

the system is in the *irregular case* where there exist a non-explosive solution for an arbitrary ε_{t+1}. In this case, the mds ε_{t+1}, which is called a *sunspot*, and the solution is called a *stationary sunspot equilibrium* (SSE).[9] The equilibrium of the economy is indeterminate given an initial state of w_t.

In the irregular case, if a quadratic equation associated with the system has real roots, the GF representation (4) can be transformed to *common factor* (CF) representations (Evans and McGough, 2005c). In our model, the associated quadratic equation of the system (i.e., $\beta b^2 - b = 0$) has real roots $b = \left\{0, \beta^{-1}\right\}$ so that for the root β^{-1}, Eq. (4) is transformed to a CF representation:

$$\begin{aligned} y_t &= (I_n - \beta\Phi)^{-1} \gamma w_t + \xi_t, && (6) \\ \xi_t &= \beta^{-1} \xi_{t-1} + \tilde{\varepsilon}_t, && (7) \end{aligned}$$

where $\tilde{\varepsilon}_t = \varepsilon_t - (I_n - \beta\Phi)^{-1} \gamma w_t + \left((I_n - \beta\Phi)^{-1} - I_n\right) \beta^{-1} \gamma w_{t-1}$, and ξ_t and $\tilde{\varepsilon}_t$ are also martingale difference sequences.[10] We assume $E\left(w_t \tilde{\varepsilon}_t\right) =$

[9] McGough and Nakagawa (2016) and Branch, McGough, and Zhu (2017) finds that sunspot equilibria are stable under adaptive learning in the parameter region where equilibrium is determinate under rational expectations.

[10] The SEEs (4) also have the other CF representation for the root $b = 0$, but the instability of this representation is trivial and not discussed here.

$E(w_t\xi_t) = 0$. If agents make an arbitrary forecast error ε_t leading to ξ_t, there exists a stationary sunspot equilibrium for ξ_t.

Hereafter, we assume that agents believe the autocorrelated sunspot (7) under adaptive learning and focus on the stability of sunspot equilibria driven by ξ_t.

3 Sunspot equilibria under perfect information

This section shows a benchmark analysis on the stability of sunspot equilibria under perfect information. We review the conventional wisdom that sunspot equilibria are always unstable under adaptive learning. We also confirm that if the steady state of the economy is observable, the result is modified. The reader who is familiar with the benchmark results can skip this section.

3.1 Adaptive learning

Suppose that agents do not have the knowledge of the economic structure enough to coordinate on rational expectations. An alternative formation of expectations is to form their forecasts $E_t^* y_{t+1}$ by estimating econometric models with all available data up to time t, $\{y_s, w_s', \xi_s\}_{s=1}^t$.[11] Following the methodology of the learning literature, this paper assumes that agents know the functional form of sunspot equilibria of CF representation (6) and estimate the perceived law of motion (PLM) of the same form:

$$y_t = a + cw_t + d\xi_t + e_t, \tag{8}$$

where a is a constant term, c' is the n-vector of coefficients for w_t, and e_t is an error term that is perceived to be white noise.[12] For simplicity, we assume that agents believe the same sunspot shock ξ_t in Eq. (7).

[11] Our analytical results are independent of whether a contemporaneous endogenous variable y_t is not used to form the forecast $E_t^* y_{t+1}$ because the forecast (9) is not determined by y_t.

[12] Although agents could specify the PLM of GF representation, discussions about sunspot equilibria of GF representation are skipped in this paper because they are found to be unstable under adaptive learning (see Evans and McGough, 2005c).

Using the estimated parameters $\phi' \equiv (a, c, d)$, agents form the forecast:

$$E_t^* y_{t+1} = a + c\Phi w_t + d\beta^{-1}\xi_t. \tag{9}$$

The actual law of motion (ALM) of the economy is determined by incorporating Eq. (9) into Eq. (1) as

$$y_t = \beta a + (\beta c\Phi + \gamma) w_t + d\xi_t. \tag{10}$$

3.2 Fixed point

We assume that in real-time learning, the parameters $\phi_t' = (a_t, c_t, d_t)$ estimated at time t are optimal linear projections of y_{t-1} on $z_{t-1}' \equiv \left(1, w_{t-1}', \xi_{t-1}\right)$ that satisfy the following least-squares orthogonality condition:

$$Ez_{t-1}\left(y_{t-1} - \phi_t' z_{t-1}\right) = 0.$$

The local dynamics of ϕ' are governed by the associated ordinary differential equation (ODE) (see Evans and Honkapohja, 2001, chapter 6):

$$\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{11}$$

where τ denotes notional time and $T(\phi)$ is the mapping from the PLM to the ALM:

$$T(\phi) \equiv \begin{pmatrix} T_a(a) & T_c(c) & T_d(d) \end{pmatrix} = \begin{pmatrix} \beta a & \beta c\Phi + \gamma & d \end{pmatrix}.$$

The fixed point of the ODE $\bar{\phi}' = \left(\bar{a}, \bar{c}, \bar{d}\right)$ is

$$\bar{a} = 0, \quad \bar{c} = (I_n - \beta\Phi)^{-1}\gamma, \quad \bar{d} = \text{arbitrary}. \tag{12}$$

If the ODE (11) is locally asymptotically stable around the fixed point $\bar{\phi}$, the parameters ϕ_t under real-time learning converge to the fixed point, and the economy is determined at the process (10) with the fixed point $\bar{\phi}$. $\bar{d}$ is arbitrary so that there can exist stable sunspot equilibria under adaptive learning, which are equivalent with sunspot REEs of CF representation (6)–(7). Thus,

Lemma 1 *Sunspot equilibria (12) under adaptive learning are stationary if and only if Eq. (5).*

Note that the fixed point of the constant term a corresponds to the steady state of y_t.[13] If the steady state is observable for agents, they do not have to estimate this term, but they may fix it at the value of the steady state. It is known that excluding the estimation of the constant term could affect the stability of an equilibrium (see Bullard and Mitra, 2002; Ji and Xiao, 2017). In addition, standard macroeconomic models assume that the knowledge of the steady state is held by agents.[14] To obtain robust implications to prevent the emergence of sunspot equilibria, our paper will examine the stability of the equilibria in both cases of the unobservable and observable steady state.

3.3 Stability conditions

The ODE is locally stable if and only if the eigenvalues of the Jacobians regarding (a, c),

$$\begin{aligned} D\left(T_a\left(a\right) - a\right) &= \beta - 1, \\ D\left(T_c\left(c\right) - c\right) &= \Phi \otimes \beta - I_n, \end{aligned}$$

have negative real parts:[15]

$$\beta < 1, \tag{13}$$

$$\beta\lambda\left[\Phi\right] < 1. \tag{14}$$

The notation $\lambda\left[X\right]$ denotes the largest value of the real parts of the eigenvalues of the matrix X and will be used throughout this paper. Eqs.

[13] If the mean of v_t is nonzero, the fixed point of the constant term of the PLM should be different from the steady state of y_t. This case does not affect our analytical results about the stability.

[14] For example, in New Keyensian models, the steady state of ouput is assumed to be observable so that the central bank can control the nominal interest rate in response to output gap, which is the difference between actual output and the steady state of output. In addition, the steady state of the inflation rate is determined by the central bank as the inflation target.

[15] Actually, the ODE can also be asymptotically stable if they have one or more zero real parts of eigenvalues, which are ruled out in this paper as nongeneric cases.

(13)–(14) are stability conditions of sunspot equilibria under adaptive learning. The stability condition is represented by only Eq. (13) as $\lambda[\Phi] = \max\{\varphi_i\}_{1\leq i\leq n} < 1$.

Notice that Eq. (13) is the condition when the steady state of y_t is unobservable for agents. If the steady state is observable so that agents may fix the constant term a at the value of the steady state, Eq. (14) is the unique stability condition.

Combined with the stationary condition (5), the stability conditions of stationary sunspot equilibria with perfect information are given as follows:

Proposition 1 *Consider the system (1)–(2) with the stationary condition (5). When the steady state is unobservable, stationary sunspot equilibria of CF representation (12) are always unstable under adaptive learning with perfect information. When the steady state is observable, they are locally stable if and only if*

$$1 < \beta < \lambda[\Phi]^{-1}. \tag{15}$$

The left-hand-side of Eq. (15) comes from the stationary condition, and the right-hand-side comes from the stability condition.

The proposition confirms the conventional result in the learning literature (see Duffy and Xiao, 2007; Ji and Xiao, 2017). It is commonly assumed that agents specify PLMs with constant terms, which premises the steady state to be unobservable. It leads to the instability of sunspot equilibria in calibrated business cycle models.

This stability puzzle is partly resolved under the observable steady state, but the stability condition (15) might not be consistent with calibrated parameters in business cycle models or empirical findings. For example, if the largest autocorrelation coefficient of $\{w_{it}\}_{i=1}^{n}$ is so large as 0.9, then the condition (15) is $1 < \beta < 1.11\cdots$, which seem too narrow. In such a case, the stability puzzle cannot be resolved only by the observable steady state.[16]

[16]Section 6.4 will show the possibility of this case in calibrated NK models.

4 Sunspot equilibria with imperfect information

In what follows, we relax the assumption of perfect information by introducing imperfect information about exogenous variables.

4.1 Private Information

To cover a variety of the information structures of exogenous variables, this paper introduces the following type of private information, which is incorporated in Nakagawa (2015) that follows the manner of Marcet and Sargent (1989a):

Assumption 1 *For each $i \in \{1, ..., n\}$, the evolution of the exogenous variable $\{w_{is}\}_{s=1}^{t}$ is observable for agents of type i and unobservable for agents of other types.*

That is, each exogenous variable of the economy is privately observable for a part of agents. Implicitly, it is also assumed that different types cannot share any information of unobservable variables. Then, agents of type i recognize the stochastic characteristics of observable variable w_{it}, but they recognize nothing about the unobservable variables $\{w_{jt}\}_{j=1,j\neq i}^{n}$.[17] For simplicity, the population of each type is assumed to be the same at $\frac{1}{n}$.

This framework is a version of the *perpetually and symmetrically uninformed* model, which is originally developed by Lucas (1972), Townsend (1983), and Pearlman and Sargent (2005), and first analyzed under adaptive learning by Marcet and Sargent (1989a). The Marcet and Sargent

[17]For example, the stochastic distributions of $\{w_j\}_{j\neq i}^{n}$, the correlations of exogenous variables $\{\rho_{ij}\}_{i,j=1,i\neq j}^{n}$, and the number n of exogenous variables are unobservable for other types of agents. If the correlations and the quantity of exogenous variables were the common knowledge, agents could use those information in adaptive learning, which should be different from the form described in this paper. This type of private information describes a feature of the information of idiosyncratic fundamental shocks at a microeconomic level: for example, a preference shock possessed by a household (see Allen and Gale, 2004) and the profitability of a borrower in a financial market (see Stiglitz and Weiss, 1981).

(1989a)'s model incorporates the information imperfection of not only exogenous variables, but also endogenous variables. Assumption 1 allows only the former imperfection to obtain analytical results of the effect of information imperfection.

The advantage of our framework is that it covers not only private information, but also a variety of the information structures of exogenous variables. In this framework, the number n of exogenous variables defines the degree of *limitation* in the information set of each agent; the larger n is, the more limited each information set is relative to the full one.[18] The parameter $1-\rho_{ij}$ (or the correlation ρ_{ij}) of the two exogenous variables $\{w_{it}, w_{jt}\}$ for each $i, j \in \{1, ..., n\}$ defines the degree of the *heterogeneity* (or *homogeneity*) in the information sets of types i and j; the smaller ρ_{ij} is, the more heterogeneous both information sets are.[19] If $n = 1$ (no limitation) or $\rho_{ij} = 1$ (and hence $\varphi_i = \varphi_j$ and $\frac{w_{it}}{\sigma_{ii}} = \frac{w_{jt}}{\sigma_{jj}}$) for all i, j (no heterogeneity), the information sets of all types are reduced to the full information set in Section 3, and the analysis of this section covers the benchmark analysis of Section 3 as a special case. In addition, a broad class of the information imperfection of exogenous variables considered in the literature is reproduced by accommodating the characteristics of $\{w_{it}\}_{i=1}^{n}$ (here, n, $\{\rho_{ij}\}_{i,j=1}^{n}$, and $\{\varphi_i, \sigma_{ii}\}_{i=1}^{n}$).[20] Thus, the effects of the private information that will be found in this paper are robust in those other information structures.

4.2 Adaptive learning

Let us describe adaptive learning by the agent of type i with the imperfect information set $\{y_s, w_{is}, \xi_s\}_{s=1}^{t}$, which is limited and different from the information sets held by other types in terms of $\{w_{it}\}_{i=1}^{n}$. We assumes

[18]For private information to exist for each type of agents, the number of unobservable variables $(n-1)$ must be greater than the number of endogenous variables (m). In the case where $m = 1$, $n > 2$ must be satisfied.

[19]Although the degree of heterogeneity in the information sets of all types cannot be represented by a single scalar measure, we will say that the degree of heterogeneity in the information sets of all types increases if ρ_{ij} falls for at least one (i, j) and falls or remains the same for every (i, j).

[20]Nakagawa (2015) demonstrates that this framework reproduces, as examples, partial information sets seen in the RPE literature, asymmetric information sets, and different populations of the types of agents.

that the agent recognizes the correct form of sunspot equilibria (8), and that the agent is constrained to specify an underparameterized PLM:

$$y_t = a_i + c_i w_{it} + d_i \xi_t + e_{it}. \tag{16}$$

For simplicity, we assume that agents of all types believe the sunspot variable ξ_t in Eq. (7). Using the estimated parameters, the forecast of agent i at time t is formed as

$$E^*_{it} y_{t+1} = a_i + c_i \varphi_i w_{it} + d_i \beta^{-1} \xi_t, \tag{17}$$

where E^*_{it} is the operator of expectations formed by type i at time t.

Nakagawa (2015) names this learning *heterogeneously misspecified (HM)* learning. The PLM and the forecast of each type is heterogeneously misspecified to the same degree as the degrees of information imperfection. The *degree of misspecification* in the PLM/forecast of each type is specified one-to-one by the degree of limitation of each information set (that is, n). The *degree of heterogeneity* in the PLMs/forecasts of types i and j is specified one-to-one by the degree of heterogeneity of information sets of both types (that is, $1 - \rho_{ij}$).

The forecast $E^*_t y_{t+1}$ in Eq. (1) is determined by the average of the forecasts of all types of the form (17). Following the same populations of different types, let us assume that forecasts of different types $\{E^*_{it} y_{t+1}\}_{i=1}^n$ have equal contributions to the dynamics of the economy. Then, the forecast $E^*_t y_{t+1}$ is formulated to have the same form as the one under perfect information (9):

$$E^*_t y_{t+1} = a + c\Phi w_t + d\beta^{-1} \xi_t, \tag{18}$$

where E^*_t is the operator of the average of heterogeneous forecasts $E^*_t = \frac{1}{n} \sum_{i=1}^n E^*_{it}$, and $a \equiv \frac{1}{n} \sum_{i=1}^n a_i$, $d \equiv \frac{1}{n} \sum_{i=1}^n d_i$ are the averages of the constant terms and coefficients for all types, and $c \equiv \frac{1}{n} (c_1, \cdots, c_n)$ is an $1 \times n$ matrix that combines the coefficients $\{c_i\}_{i=1}^n$ of the PLMs of the form (16) for all types and multiplies them by $\frac{1}{n}$.

The ALM is obtained by substituting Eq. (17) into the system (1)–(2):

$$y_t = \beta a + (\beta c \Phi + \gamma) w_t + d\xi_t. \tag{19}$$

4.3 Fixed point

As in Section 3, the parameters of type i, $\phi'_{it} = (a_{it}, c_{it}, d_{it})$ are assumed to be optimal linear projections of y_{t-1} on $z'_{i,t-1} \equiv \left(1, w_{i,t-1}, \xi_{i,t-1}\right)$ that satisfy the following least-squares orthogonality condition,

$$Ez_{i,t-1}\left(y_{t-1} - \phi'_{it} z_{i,t-1}\right) = 0,$$

such that the misspecification in the PLM (16) is not detected.

The local dynamics of ϕ_i in real-time learning is inferred from the stochastic recursive algorithms of ϕ_i formulated by the PLM (16) and the ALM (19). As a result, the dynamics of the parameters of all types are represented by the dynamics of the aggregate parameters $\phi' = (a, c, d)$, which are governed by the following ODE:

$$\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{20}$$

where

$$T(\phi) \equiv \begin{pmatrix} T_a(a) & T_c(c) & T_d(d) \end{pmatrix} = \begin{pmatrix} \beta a & (\beta c \Phi + \gamma)\left(\frac{1}{n}\Psi\right) & d \end{pmatrix},$$

and

$$\Psi \equiv diag\left(\sigma_{ii}\right)_{1 \leq i \leq n} \cdot \Gamma \cdot diag\left(\sigma_{ii}\right)^{-1}_{1 \leq i \leq n}.$$

The derivation of the ODE is shown in Appendix B. Mapping $T(\phi)$ provides the coefficients of the forecasts of y_t updated by agents of all types.

The fixed points $\bar{\phi}' = \left(\bar{a}, \bar{c}, \bar{d}\right)$ are

$$\bar{a} = 0, \quad \bar{c} = (\beta \bar{c} \Phi + \gamma)\left(\frac{1}{n}\Psi\right), \quad \bar{d} = \text{arbitrary}. \tag{21}$$

If the ODE is locally stable at the fixed point $\bar{\phi}$, the aggregate parameters ϕ_t under real-time learning converge to the fixed point, and the economy is determined with Eqs. (19) and (21). The fixed point $\bar{d}$ is arbitrary so that the sunspot shock ξ_t drives sunspot equilibria under imperfect information as well as the ones under perfect information, and even under information imperfection, Eq. (5) is the stationary condition for sunspot equilibria.

The effect of information imperfection on the fixed point is represented by $\left(\frac{1}{n}\Psi\right)$. If there exists no imperfect information (that is, $n = 1$ or

$\{\rho_{ij} = 1\}_{i,j=1}^{n}$ for all i, j), all the specifications in this section is reduced to the ones under perfect information in Section 3.

Let us define the equilibria (21) as sunspot *heterogeneous misspecification equilibria* (hereafter, sunspot *HMEs*).

Definition 1 *The sunspot HME is a stochastic process for $\{y_t\}_{t=0}^{\infty}$ following the system (1)–(2) given that $\{E_t^* y_{t+1}\}_{t=0}^{\infty}$ is the average of the forecasts formed by the PLMs of the form (16) for all i with the parameters $\{\phi_i' = (a_i, c_i, d_i)\}_{i=1}^{n}$ determined at the fixed point (21) of the ODE (20).*

4.4 Stability conditions

The stability of sunspot HMEs is subject to whether the aggregate parameters $\phi' \equiv (a, c, d)$ converge to the fixed point. They are locally stable if and only if the eigenvalues of the Jacobians of the ODE (20) regarding (a, c),

$$\begin{aligned} D\left(T_a\left(a\right) - a\right) &= \beta - 1, \\ D\left(T_c\left(c\right) - c\right) &= \left(\Phi\left(\frac{1}{n}\Psi\right)\right)' \otimes \beta - I_n, \end{aligned}$$

have negative real parts:

$$\begin{aligned} \beta &< 1, && (22) \\ \beta\Lambda &< 1, && (23) \end{aligned}$$

where

$$\Lambda \equiv \lambda\left[\Phi\left(\frac{1}{n}\Psi\right)\right] = \lambda\left[\Phi\left(\frac{1}{n}\Gamma\right)\right] < 1.$$

The effect of information imperfection on the stability is represented by the parameter Λ. Nakagawa (2015, Lemma 1, Remark 1) proves that the parameter Λ depends on the degrees of information imperfection $\left(n, \{\rho_{ij}\}_{i,j=1}^{n}\right)$ as follows:

Lemma 2 *For each $n \geq 1$ and $i, j \in \{1, \cdots, n\}$,*

1. *all eigenvalues of* $\Phi\left(\frac{1}{n}\Gamma\right)$ *are real and exist in the interval* $[0,1)$*;*

2. $\Lambda \leq \lambda[\Phi]$ *with equality iff* $n = 1$ *or* $\rho_{ij} = 1$ *for all* i, j*.*

3. $\frac{d\Lambda}{d\varphi_i} \geq 0$*;*

4. $\frac{d\Lambda}{d(1-\rho_{ij})} \leq 0$*;*

5. $\frac{d\Lambda}{dn} \leq 0$ *with the equality iff* $\rho = 1$ *or* $\varphi = 0$*, if*

$$\varphi_i = \varphi \in [0,1) \text{ for all } i, \tag{24}$$

$$\rho_{ij} = \rho \in [0,1] \text{ for all } i, j \text{ and } i \neq j. \tag{25}$$

Lemma 2.1 means that $0 \leq \Lambda < 1$. Then, if the steady state is unobservable, the stability condition is represented by Eq. (22). Otherwise, the stability condition is represented by Eq. (23). Lemma 2.2 indicates that if there exists no information imperfection (that is, $n = 1$ or $\left\{\rho_{ij} = 1\right\}_{i,j=1}^{n}$ for all i, j), the stability condition (23) is reduced to the one under perfect information (14).

Combined with the stationary condition (5), the stability conditions of stationary sunspot equilibria under imperfect information are given as follows:

Proposition 2 *Consider the system (1)–(2) satisfying the stationary condition (5). When the steady state is unobservable, stationary sunspot equilibria (21) are always unstable under adaptive learning with imperfect information. When the steady state is observable, they are locally stable if and only if*

$$1 < \beta < \Lambda^{-1}. \tag{26}$$

Note that under no imperfect information, the proposition is reduced to Proposition 1.

The proposition suggests that in a model with the unobservable steady state, sunspot equilibria are always unstable under adaptive learning. This result is independent of the existence of imperfect information.

When the steady state is observable, the stability of sunspot equilibria is affected by imperfect information as follows:

Corollary 1 *Suppose that the steady state is observable. For each* $n \geq 1$ *and* $i, j \in \{1, \cdots, n\}$*, the stability condition (26) is unchanged or relaxed upwards by an increase in the degree of heterogeneity* $1 - \rho_{ij}$ *in the information sets of types* i *and* j *for each* i, j*. The condition is unchanged or relaxed upwards by an increase in the degree of limitation* n *of the information set of each type if exogenous variables have the same stochastic characteristics as Eqs. (24)–(25).*

The proof is trivial by Lemmas 2.4 & 2.5. That is, the heterogeneity and limitation of information sets have nonnegative effects on the stability of sunspot equilibria so that they can be stable in the economy with a large positive feedback of expectations.

These results clarify a channel through which self-fulfilling business cycle fluctuations happens under the imperfect information of fundamental shocks. Similar mechanisms have been demonstrated in a bunch of the rational expectations literature (e.g., Jermann and Quadrini, 2012; Benhabib and Wang, 2013; Gertler and Kiyotaki, 2015), where imperfect information gives an positive effect on the stationarity of sunspot equilibria. Our results show that imperfect information has an additional effect of improving the stability of those equilibria. Thus, under either framework of expectations formation, the economy with highly imperfect information is inclined to experience self-fulfilling fluctuations.

Our results also provide a policy implication that even if a particular policy prevented self-fulfilling fluctuations, it might be ineffective under imperfect information. In such a situation, the government should modify the policy to increase the feedback of expectations (β) to prevent them.

4.5 Discussion

The nonnegative effect of imperfect information is intuitive. The effect of heterogeneity results from the reductions in the extent of the updating of the parameters $\{c_i\}_{i=1}^n$. Suppose that an exogenous variable w_{it} evolves. Without heterogeneity, all types of agents update $\{c_i\}_{i=1}^n$ to the same degree. In contrast, in the presence of heterogeneity because of the imperfect correlations of $\{w_{it}\}_{i=1}^n$, the types except type i do not update $\{c_j\}_{j\neq i}^n$ to the same degree as the updating of c_i. These reductions make the aggregate forecast $E_t^* y_{t+1}$ and the economy y_t less responsive to the evolution of w_t so that they produce a feedback that helps the parameters to converge. Similarly, the effect of misspecification results from the reductions in the extent of the updating of $\{c_i\}_{i=1}^n$. Without misspecification ($n = 1$), agents update $\{c_i\}_{i=1}^n$ in response to the evolution of each variable. In the presence of misspecification ($n > 1$), agents update only if their observable variables evolve.[21]

This mechanism is illustrated in brief calibrations. Assume $\beta = 1.5$ and $E\left(\xi_t^2\right) = 1$. Suppose the same stochastic characteristics of exogenous variables in Eqs. (24) and (25): $\varphi_i = 0.9$, $E\left(w_{it}^2\right) = 1$, $\gamma_i = 1$ for all i and $\rho_{ij} = \rho$ for all $i \neq j$.

Figure 1 shows the calibrations of the updating of the parameter c_{it} and the adjustment of endogenous variable y_t in response to a one-time evolution of $w_{1,0} = 1$ under different degrees of heterogeneity of agents information sets. The initial values of parameters (a, c, d) are set at $(0, 0, 1)$. The degree of limitation (n) is fixed at 10. The stability condition under the observable steady state (26) is $\rho < 0.71$. In the figure, the response of y_t to the evolution of w_1 is reduced as the degree of heterogeneity $(1 - \rho)$ increases. This reduction makes the updating of c_t sluggish so as to converge. On the other hand, if $1 - \rho$ is lower than the threshold, the response of y_t and the updating of c_t are so large as to explode.

Figure 2 shows calibrations under different degrees of limitation of agents' information sets. Here, $1-\rho$ is fixed at 0.3, and the other settings are unchanged. The stability condition under the observable steady state is $n > 7.36$. In the figure, the response of y_t is also reduced as n increases. It makes the updating of c_t sluggish enough to converge.

These effect works even if the steady state is observable for *a small proportion* of agents, although its observability for *all* agents has been assumed for simplicity of our analysis. If the proportion of agents who observe the steady state is assumed to be $p \in [0, 1]$, the stability condition of the constant term (22) is modified to $\beta (1 - p) < 1$ and the condition of c (28) is unchanged. If $p > 1 - \lambda [\Phi]$, the stability condition of sunspot equilibria is represented by Eq. (28), which is affected by imperfect in-

[21]Note that even if $n \to \infty$ such that each w_{it} has a negligible impact on the economy, $\{c_i\}_{i=1}^n$ do not necessarily converge to zero, but continue to be updated because $\{w_{it}\}_{i=1}^n$ are correlated.

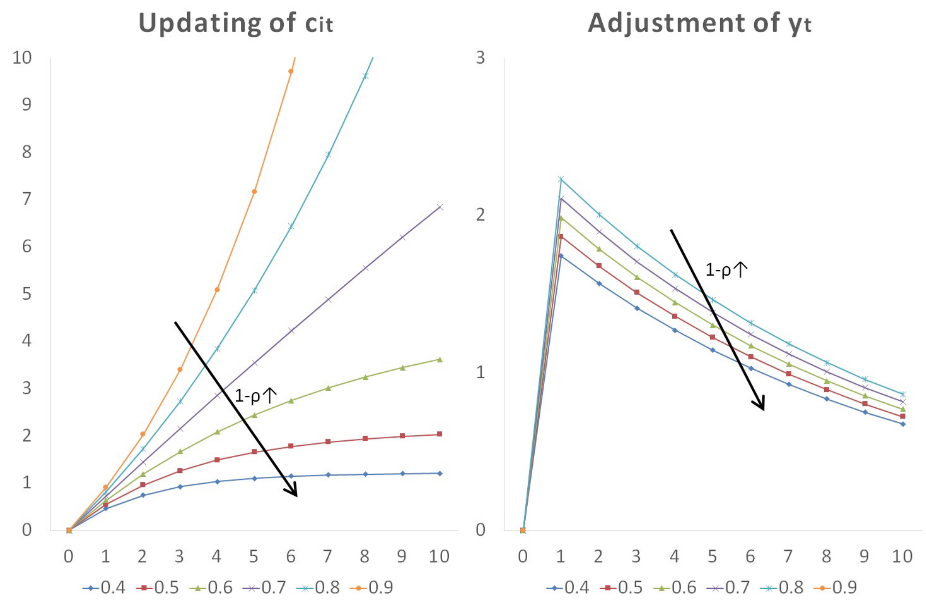

Figure 1: The updating of the parameter c_{it} and the adjustment processes of y_t under different digrees of information heterogeneity

formation. If, for example, $\varphi_i = 0.9$ for all i, then $p > 1 - \lambda[\Phi] = 0.1$: the information imperfection is effective if the steady state is observable for more than ten percent of agents.

5 Stability conditions in a multivariate model

The previous results in the univariate model are easily generalized in a multivariate model. Let us consider the following $m-$variate system:

$$y_t = BE_t^* y_{t+1} + Cw_t, \quad (27)$$

$$w_t = \Phi w_{t-1} + v_t, \quad (28)$$

where y_t is an $m \times 1$ matrix, B is an $m \times m$ matrix, C is an $m \times n$ matrix, and other things are exactly the same as before. $\det(B) \neq 0$ is assumed.

Let us also assume the system (27)–(28) to have the positive feedback of expectations, that is, the eigenvalues of B have all positive real parts.

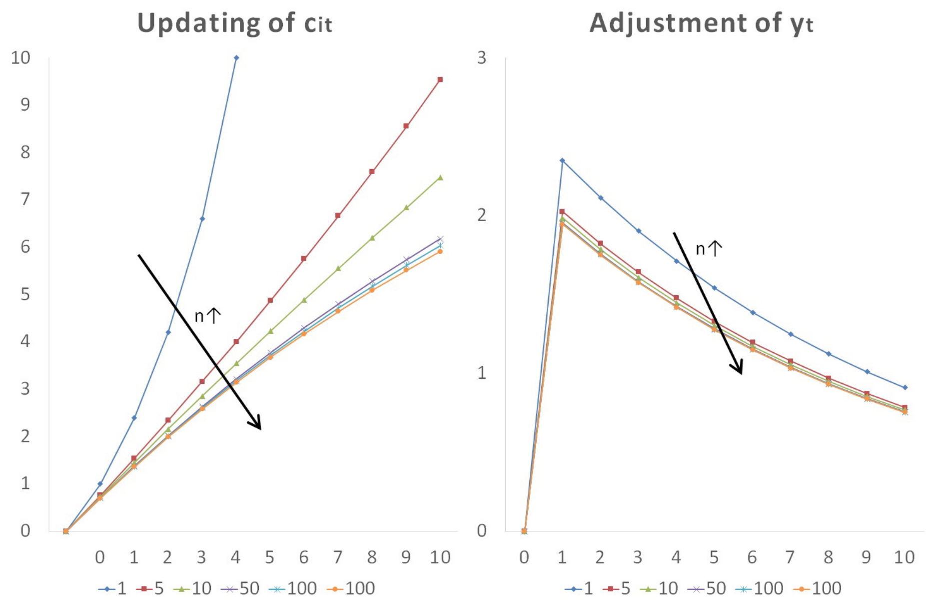

Figure 2: The updating of the parameter c_{it} and the adjustment processes of y_t under different degrees of information limitation

This is represented by[22]

$$\lambda\left[-B\right] < 0. \tag{29}$$

Agents of type i are assumed to specify the same form of PLM as Eq. (16); here a_i is an $m \times 1$ matrix, c_i is an $m \times 1$ matrix, d_i is an $m \times 1$ matrix, and e_{it} is an $m \times 1$ matrix.[23] We assume that agents believe a sunspot shock ξ_t

$$\xi_t = \theta^{-1}\xi_{t-1} + \tilde{\varepsilon}_t,$$

where θ denotes either of the nonzero real eigenvalues of matrix B, if any. The rest of the mechanism is the same as in Section 4.2. Then, the ALM is obtained as

$$y_t = Ba + \left(Bc\Phi + C\right)w_t + B\theta^{-1}d\xi_t,$$

where $a \equiv \frac{1}{n}\sum_{i=1}^{n} a_i$, $c \equiv \frac{1}{n}\left(c_1, \cdots, c_n\right)$, and $d \equiv \frac{1}{n}\sum_{i=1}^{n} d_i$.

[22]Note that $\lambda\left[-\beta\right] \neq -\lambda\left[\beta\right]$.

[23]The order two or more indeterminacy considered in Evans and McGough (2005c) is not discussed in our paper.

In this case, the associated ODE of $\phi' = (a, c, d)$ is represented by the same form as the ODE (20); here

$$T(\phi) \equiv \begin{pmatrix} T_a(a) & T_c(c) & T_d(d) \end{pmatrix} = \begin{pmatrix} Ba & (Bc\Phi + C)\left(\frac{1}{n}\Psi\right) & B\theta^{-1}d \end{pmatrix}.$$

Then, the fixed point $\bar{\phi}' = \left(\bar{a}, \bar{c}, \bar{d}\right)$ is obtained as

$$\bar{a} = 0, \quad \bar{c} = (B\bar{c}\Phi + C)\left(\frac{1}{n}\Psi\right), \quad \bar{d} = s\hat{d}, \tag{30}$$

where $\bar{d}$ is the product of an arbitrary real constant s and the real eigenvector $\hat{d}$ corresponding to the eigenvalue θ, and there exist a continuum of the fixed point $\bar{d}$.

Corresponding to Lemma 1 in the univariate model,

Lemma 3 *In the multivariate system (27)–(28), there exist sunspot equilibria of CF representation (30) if and only if either of the eigenvalues of matrix B is real. If any, those equilibria are stationary if and only if either of the real eigenvalues is outside the unit circle.*

For sunspot equilibria to be locally stable under adaptive learning, the Jacobians of the associated ODE regarding (a, c),

$$\begin{aligned} D(T_a(a) - a) &= B - I_m, \\ D(T_c(c) - c) &= \left(\Phi\left(\frac{1}{n}\Psi\right)\right)' \otimes B - I_{mn}, \end{aligned}$$

must have all negative real parts of eigenvalues, that is,

$$\lambda[B] < 1, \tag{31}$$

$$\lambda[B]\,\Lambda < 1. \tag{32}$$

Therefore, the stability condition of sunspot equilibria of CF representation is given as follows:

Proposition 3 *Consider the system (27)–(28) with the positive feedback of expectations (29) and the real and stationary conditions in Lemma 3. When the steady state is unobservable, those equilibria are always locally unstable under adaptive learning. When the steady state is observable, they are locally stable if and only if Eq. (32).*

Note that the instability under the unobservable steady state is trivial. Matrix B must satisfy Eqs. (29) and (31); that is, the real parts of the eigenvalues of B must be all inside the unit circle. This contradicts with the conditions in Lemma 3.

6 Application to a New Keynesian model

This section shares stability conditions for sunspot equilibria in a basic New Keynesian model:

$$\begin{aligned} x_t &= -\alpha \left(i_t - E_t^* \pi_{t+1}\right) + E_t^* x_{t+1}, && (33) \\ \pi_t &= \kappa x_t + \eta E_t^* \pi_{t+1}. && (34) \end{aligned}$$

The model has three endogenous variables: output gap x_t, the inflation rate π_t, and the nominal interest rate i_t. Eq. (33) is a log-linearized intertemporal Euler equation that is derived from the households' optimal choice of consumption. Eq. (34) is a Phillips curve with the forward-looking component that is derived from the optimizing behavior of monopolistically competitive firms with Calvo price setting. $\alpha > 0$, $\kappa > 0$, and $0 < \eta < 1$ are assumed. Demand and supply shocks are omitted for simplicity of the analysis.[24] Throughout this section, the observable steady state is assumed because sunspot equilibria are unstable if the steady state is unobservable.

We consider three types of nominal interest rate rules of the central bank that are popular in the literature: a *current*-looking nominal interest rate rule,

$$i_t = \phi_\pi \pi_t + \phi_x x_t + w_t, \tag{35}$$

a *forward*-looking rule,

$$i_t = \phi_\pi E_t^* \pi_{t+1} + \phi_x E_t^* x_{t+1} + w_t, \tag{36}$$

and a *semi-forward*-looking rule,

$$i_t = \phi_\pi E_t^* \pi_{t+1} + \phi_x x_t + w_t, \tag{37}$$

[24] Our analytical results are robust to the incorpolation of demand or supply shocks and their imperfect information. In that case, the NK model should have a different form, which complicates our analysis.

where w_t is a monetary policy shock. The parameters ϕ_π and ϕ_x are controlled by the central bank, and $\phi_\pi, \phi_x \geq 0$ are assumed.

Under rational expectations,

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) > 0 \tag{38}$$

is the sufficient and necessary condition for determinate REE under the current-looking rule and a necessary condition under the forward- and semi-forward-looking rule (see Bullard and Mitra (2002, Propositions 1 & 4) and Appendix F). Eq. (38) is called the *Taylor principle*, the importance of which is emphasized by Woodford (2003) and others. In later analysis, stability conditions of sunspot equilibria will be not only obtained, but also compared with the Taylor principle.

To introduce the imperfect information of exogenous variables, the policy shock w_t is assumed to be the aggregation of individual monetary policy shocks: $w_t \equiv \sum_{i=1}^n w_{it}$. These shocks describe, for example, the preference shocks of different policy board members of the central bank.[25] The shock w_{it} for each $i \in \{1, \cdots, n\}$ follows a persistent process: $w_{it} = \varphi_i w_{i,t-1} + v_{it}$, where $0 \leq \varphi_i < 1$ and the disturbance term v_{it} has a zero mean. The correlation of w_{it} and w_{jt} is $\rho_{ij} \geq 0$ for each $i, j \in \{1, \cdots, n\}$. If w_{it} for each i is observable for all agents, they specify correctly specified PLMs and homogeneous forecasts as in Eqs. (8)–(9). If w_{it} for each i is privately observable for $\frac{1}{n}$ of agents and unobservable for other agents, they specify underparameterized PLMs and heterogeneous forecasts as in Eqs. (16)–(17). Following the assumptions given by Branch and McGough (2009), the aggregate forecasts $(E_t^* x_{t+1}, E_t^* \pi_{t+1})$ is given by the averages of the forecasts of all types $\{(E_{it}^* x_{t+1}, E_{it}^* \pi_{t+1})\}_{i=1}^n$ as in Eq. (18).[26]

[25] In this example, each policy rule may be interpreted as the average of the policy reaction functions of different members, and ϵ_t in each equation as the average of exogenous beliefs of different members (see Riboni and Ruge-Murcia, 2008).

[26] In our model, the decision rules of agents underlying the NK model (33)–(34) are identical except their expectations. Then, the NK model keeps the original form while the aggregate forecasts are replaced with the average of the forecasts of different agents (see Branch and McGough, 2009; Branch and Evans, 2011; Muto, 2011).

6.1 Forward-looking Rule

To help our understanding, let us consider first the forward-looking rule (36) and obtain the stability conditions imposed on the policy parameters. Although the NK model is multivariate, the model with the forward-looking rule can be transformed to a univariate version so that we provide the results corresponding to Section 4. After that, the results of the multivariate case will be provided.

6.1.1 Univariate version

We assume $\phi_x = \alpha^{-1}$ to transform the model into:[27]

$$\begin{aligned} x_t &= -\alpha \left(\phi_\pi - 1\right) E_t^* \pi_{t+1} - \alpha w_t, & (39) \\ \pi_t &= \left(\eta - \alpha\kappa \left(\phi_\pi - 1\right)\right) E_t^* \pi_{t+1} - \alpha\kappa w_t. & (40) \end{aligned}$$

Then, the dynamics of the economy is solely determined by Eq. (40) of a univariate equation of π_t, which determines the equilibrium of x_t in Eq. (39). The coefficients in Eq. (40) correspond to the parameters in the univariate system (1)–(2):

$$\begin{aligned} \beta &= \eta - \alpha\kappa \left(\phi_\pi - 1\right), \\ \gamma &= \alpha\kappa \left(1, \cdots, 1\right). \end{aligned}$$

The stability condition in the NK model is provided by Proposition 2:

Proposition 4 *Consider the NK model (33)–(34) and the forward-looking rule (36) with the positive feedback of expectations (29), the observable steady state, and $\phi_\pi = \alpha^{-1}$. Stationary sunspot equilibria of CF representation are locally stable under adaptive learning if and only if*

$$1 - \frac{\Lambda^{-1} - \eta}{\alpha\kappa} < \phi_\pi < 1 - \frac{1 - \eta}{\alpha\kappa}. \tag{41}$$

The right-hand-side of the condition (41) is given by the stationary condition (5). This part corresponds to the violation of the Taylor principle (38), which is consistent with the historical evidence that

[27] This simplification is seen in Evans and Honkapohja (2003), Ferrero (2007), and Muto (2011).

self-fulfilling business cycle fluctuations were synchronized with the period when central banks adopted passive nominal interest rate rules (see Introduction).[28]

The left-hand-side of the condition is given by the stability condition (23). According to Corollary 1, this part is affected by imperfect information as follows:

Corollary 2 *For each $n \geq 1$ and $i, j \in \{1, \cdots, n\}$, the stability condition (41) is unchanged or relaxed downwards by an increase in the degree of heterogeneity $1 - \rho_{ij}$ in the information sets of types i and j about the monetary policy shocks $\{w_{it}\}_{i=1}^{n}$. The condition is unchanged or relaxed downwards by an increase in the degree of limitation n of the information set of each type if $\{w_{it}\}_{i=1}^{n}$ have the same stochastic characteristics as Eqs. (24)–(25).*

That is, the imperfect information in the NK model has an effect that sunspot equilibria can be stable even if the policy rule is very passive.

Hence, if a high degree of imperfect information is considered, the implausible feature of the stability condition (41) is resolved. The condition means that sunspot equilibria are stable if the policy rule is *moderately* passive such that $\phi_\pi > 1 - \frac{\Lambda^{-1} - \eta}{\alpha\kappa}$, while they are unstable if the rule is *extremely* passive such that $0 \leq \phi_\pi \leq 1 - \frac{\Lambda^{-1} - \eta}{\alpha\kappa}$. This result is clearly inconsistent with empirical findings. However, if the degree of imperfection is high such that $\Lambda \leq \frac{1}{\eta + \alpha\kappa}$, the lower bound of the condition reaches zero so that sunspot equilibria are stable under an arbitrary passive rule of $0 \leq \phi_\pi < 1$. Thus, the imperfect information provides empirically plausible stability conditions.

The results for the policy rule are summarized as follows:

Corollary 3 *Consider the NK model in Proposition 4. When the information sets of agents are enough limited and/or heterogeneous such that $\Lambda \leq \frac{1}{\eta + \alpha\kappa}$, stationary sunspot equilibria are locally stable if and only if the Taylor principle is violated.*

[28] Without the assumptions of the observable steady state and the positive feedback restriction, the stability condition should be $\phi_\pi > 1 + \frac{1+\eta}{\alpha\kappa}$ satisfying the Taylor principle (see Evans and McGough, 2005b, Fig. 4).

The corollary reinforces the importance of the Taylor principle to prevent self-fulfilling fluctuations. In particular, the higher the degree of imperfection, the more important the central bank follows the Taylor principle.

6.1.2 Multivariate version

Let us consider the case of removing the restriction $\phi_x = -\alpha^{-1}$. The NK model (33)–(34) with the forward-looking rule (36) is represented in the following multivariate form:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = B_f E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ \kappa \end{bmatrix} w_t,$$

where

$$B_f \equiv \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 - \alpha\phi_x & -\alpha\left(\phi_\pi - 1\right) \\ 0 & \eta \end{bmatrix}.$$

Stability conditions are obtained using Proposition 3 as follows:

Proposition 5 *Consider the NK model (33)–(34) and the forward-looking rule (36) with the positive feedback of expectations (29) and the observable steady state. Stationary sunspot equilibria of CF representation exist and are locally stable under adaptive learning if and only if*

$$-\frac{(1-\Lambda)\left(1-\eta\left(1-\alpha\phi_x\right)\Lambda\right)}{\alpha\Lambda} < \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 0, \quad (42)$$

$$\phi_x \leq \alpha^{-1}, \quad (43)$$

The derivation is shown in Appendix C, and the proposition covers Proposition 4 as the special case of $\phi_x = \alpha^{-1}$. Figure 3 describes the stability conditions (42)–(43).

The features of the stability condition in the univariate case (Proposition 4 and Corollaries 2 & 3) are robust in the multivariate case. Sunspot equilibria are stable only if the policy rule violates the Taylor principle (38). As the degree of information imperfection increases so that Λ is decreases, the lower bound of the condition (42) decreases. If the degree of imperfection is very high, sunspot equilibria are locally stable if and only

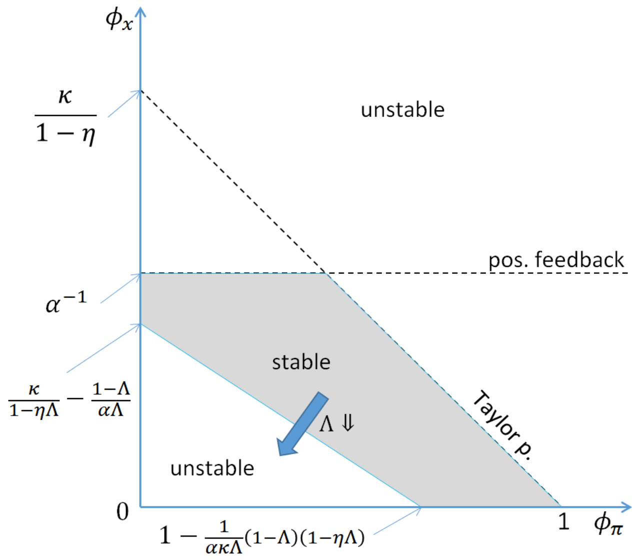

Figure 3: Stability condition under the forward-looking rule.

if the Taylor principle is violated and the condition (43) is satisfied. Under the purely forward-looking rule ($\phi_x = 0$), the violation of the Taylor principle is the necessary and sufficient condition.

6.2 Current-looking rule

The NK model (33)–(34) with the current-looking rule (35) is represented in the following form:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = B_c E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \begin{bmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} w_t,$$

where

$$B_c \equiv \begin{bmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \alpha \\ 0 & \eta \end{bmatrix}.$$

Stability conditions are obtained by Proposition 3 as follows:

Proposition 6 *Consider the NK model (33)–(34) and the current-looking rule (35) with the observable steady state. Stationary sunspot equilibria*

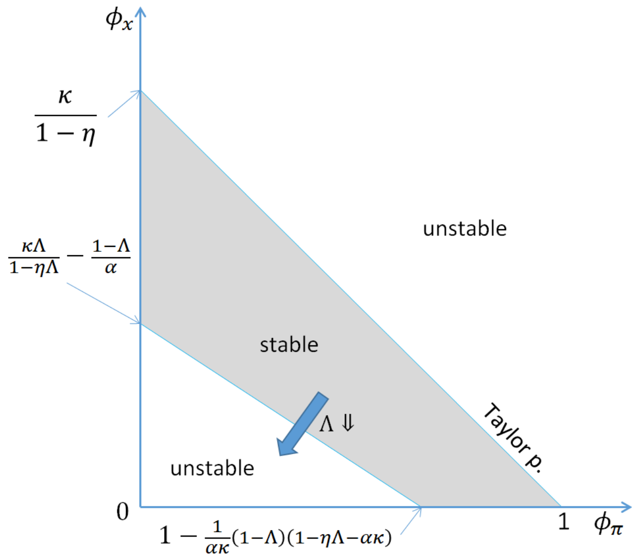

Figure 4: Stability condition under the current-looking rule.

of CF representation exist and are locally stable under adaptive learning if and only if

$$-\frac{1}{\alpha}(1-\Lambda)(1+\alpha\kappa+\alpha\eta\phi_x-\eta\Lambda) < \kappa(\phi_\pi - 1) + \phi_x(1-\eta) < 0. \quad (44)$$

The derivation is shown in Appendix D. The positive feedback of expectations (29) always holds under the sign restrictions of the structural parameters. Figure 4 describes the stability condition (44).

The stability condition under the current-looking rule has almost the same features as the ones under the forward-looking rule. The difference is that if the degree of imperfection is very high, the violation of the Taylor principle is always the necessary and sufficient for stable sunspot equilibria.

6.3 Semi-forward-looking rule

The NK model (33)–(34) with the semi-forward-looking rule (37) is represented in the following multivariate form:

$$\left[\begin{array}{c} x_t \\ \pi_t \end{array}\right] = B_{sf} E_t \left[\begin{array}{c} x_{t+1} \\ \pi_{t+1} \end{array}\right] - \alpha \left[\begin{array}{cc} 1+\alpha\phi_x & 0 \\ -\kappa & 1 \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ \kappa \end{array}\right] w_t,$$

where

$$B_{sf} \equiv \left[\begin{array}{cc} 1+\alpha\phi_x & 0 \\ -\kappa & 1 \end{array}\right]^{-1} \left[\begin{array}{cc} 1 & -\alpha\left(\phi_\pi - 1\right) \\ 0 & \eta \end{array}\right].$$

Stability conditions are obtained as follows:

Proposition 7 *Consider the NK model (33)–(34) and the semi-forward-looking rule (37) with the positive feedback restriction (29) and the observable steady state. Stationary sunspot equilibria of CF representation exist and are locally stable under adaptive learning if and only if*

$$-\frac{\left(1-\Lambda\right)\left(1+\alpha\phi_x - \eta\Lambda\right)}{\alpha\Lambda} < \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 0. \qquad (45)$$

The derivation of the proposition is shown in Appendix E.

The qualitative features of the stability condition (45) is exactly the same as the ones under the current-looking rule (44). If the stability condition (45) is described in a figure, it is similar to Figure 4.

6.4 Calibrations

Finally, let us clarify the significance of information imperfection in calibrated NK models. We calibrate stability conditions under the three types of policy rules (35)–(37).

The model is analyzed using four cases of the structural parameters (α, κ, η) in Table 1.[29] The number of monetary policy shocks $\{w_{it}\}_{i=1}^{n}$ are

[29]The structural and policy parameters are given by Table 1 of Woodford (1999), the "Pre-Volcker (Prior I)" in Table 3 of Lubik and Schorfheide (2004), Section 1.5 of McCallum and Nelson (1999), and the baseline result in Table II of Clarida, Gali, and Gertler (2000). The Lubik and Schorfheide (2004)'s parameters are estimated using a Bayesian technique, and the other parameters are used in Evans and McGough (2005b).

Table 1: Structural & policy parameters

	Structural			Policy		
	α	κ	η	Type	ϕ_π	ϕ_x
Woodford (1999)	1/0.157	0.024	0.99	c	–	–
Lubik and Schorfheide (2004)	1/1.45	0.77	0.99	c	0.77	0.17
McCallum and Nelson (1999)	0.164	0.3	0.99	f	0.87	0.17
Clarida, Gali, and Gertler (2000)	4	0.075	0.99	sf	0.83	0.27

set as $n = 10$.[30] For simplicity, we set the same stochastic characteristics of the shocks ($\varphi_i = \varphi \in [0, 1)$ for all i and $\rho_{ij} = \rho \in [0, 1)$ for all $i \neq j$). The autocorrelation φ is set as $\varphi = 0.9$.[31] Under these settings, we will treat the parameter $1 - \rho$ as not only the degree of information heterogeneity, but also the degree of information imperfection.

Table 2 shows calibrated eigenvalues of coefficient matrices (B_c, B_f, B_{sf}) using the parameters in Table 1. In Woodford (1999), policy parameters were not estimated, and the policy parameters of Lubik and Schorfheide (2004) are used. We find that all of the cases satisfy the positive feedback restriction (29) and the real and stationary conditions in Lemma 3 so that their calibrated models allow stationary sunspot equilibria of CF representation to exist.

Figure 5 calibrates the stability conditions under the current-looking rule and the degree of information heterogeneity $1 - \rho$. The parameters of Woodford (1999) and Lubik and Schorfheide (2004) are used. Panel B shows the estimates of (ϕ_π, ϕ_x) as well. The figure illustrates the analytical results in previous sections. Under the observable steady state, there emerges the stable regions that violate the Taylor principle, while there is also the implausible regions where sunspot equilibria are unstable

[30]This number refers to the numbers of the members of the US Federal Open Market Committee (12 members), the ECB Executive Board (6), and the BOJ Policy Board (9).

[31]The serial correlation of a monetary policy shock is estimated to be 0.92 by Rudebusch (2002) using US data during 1987-1999. Similar results are obtained by Consolo and Favero (2009) using the data of the pre-Volcker period. Carrillo, Feve, and Matheron (2007) estimate the serial correlation to be $0.87 \sim 0.95$ using US data during 1960-2002. Feve, Matheron, and Poilly (2007) obtain similar results using European data during 1987-2004.

Table 2: Eigenvalues

		Eigenvalues	
Woodford (1999)	B_c	1.0198	0.4412
Lubik and Schorfheide (2004)	B_c	1.1767	0.5513
McCallum and Nelson (1999)	B_f	1.0640	0.9045
Clarida, Gali, and Gertler (2000)	B_{sf}	1.0357	0.4595

if (ϕ_π, ϕ_x) are extremely small. Further, sunspot equilibria in the Lubik and Schorfheide (2004)'s model are unstable under perfect information $(1 - \rho = 0)$.

The implausibility is resolved under imperfect information. The stable regions are significantly expanded downwards as the degree of information heterogeneity $1 - \rho$ is increased. In the literature, the correlations ρ of the monetary policy shocks are not estimated so large as $\rho > 0.5$.[32] If $1 - \rho \geq 0.5$, the lower bounds of stability conditions reach the origin so that the violation of the Taylor principle (38) is the necessary and sufficient condition. This suggests that the information imperfection in agents' information sets is an important factor for calibrated NK models to exhibit empirically plausible self-fulfilling fluctuations.

Figure 6 shows the stability conditions under the forward-looking rule using the parameters of McCallum and Nelson (1999). The results are similar to the previous ones because Eq. (43) is not so restrictive as $\phi_x \leq 6.1$. In practice, under a high degree of information heterogeneity, the violation of the Taylor principle is the sufficient and necessary condition.

Figure 8 shows the stability conditions under the semi-forward-looking rule. Results are the same as the previous ones.

In total, our results suggests that calibrated NK models needs the information imperfection to obtain empirically plausible sunspot equilibria. Those models provides implausible stability conditions under perfect information, and such a implausibility is resolved by imperfect information. In addition, the Taylor principle is important to prevent stationary

[32]Bhattacharjee and Holly (2015, Fig. 1) estimate the interactions of MPC members of the Bank of England. The average of the estimated coefficients of the regressions between policy decisions of the members is 0.32, which may be close to correlations of their preferences.

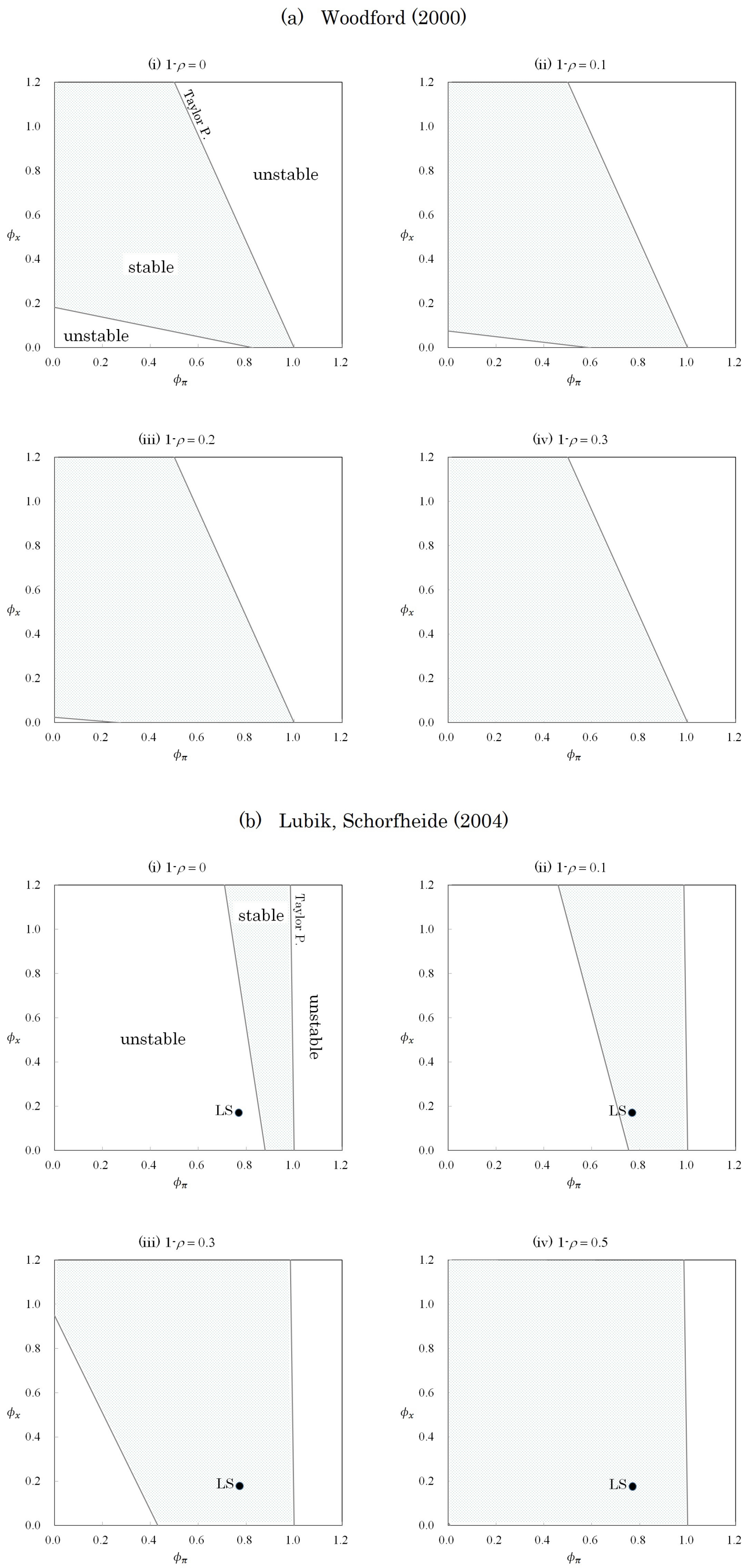

Figure 5: Stability conditions under different digrees of information heterogeneity (current-looking rule)

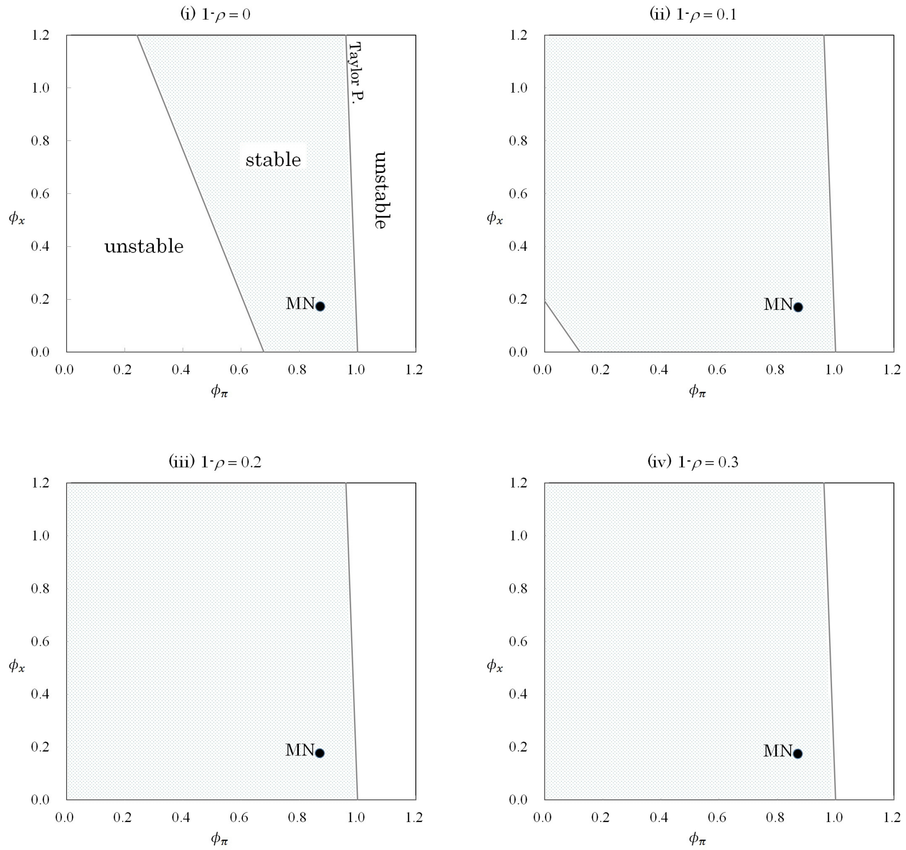

Figure 6: Stability conditions under the forward-looking rule (Parameters: McCallum and Nelson, 1999).

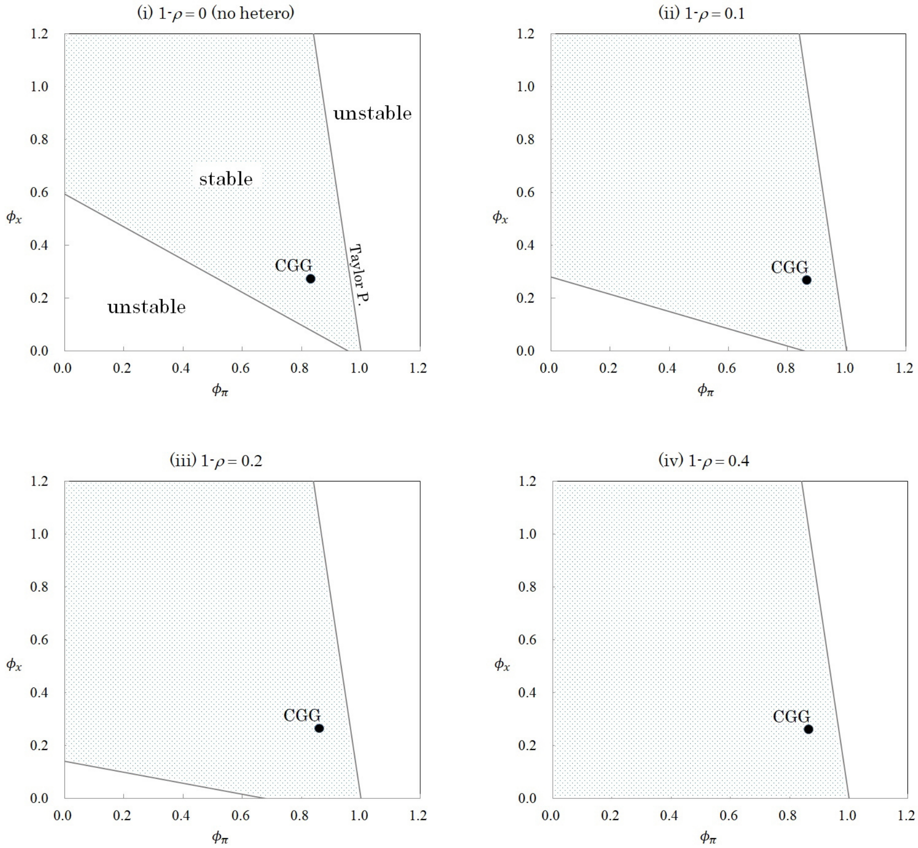

Figure 7: Stability conditions under the semi-forward-looking rule (Parameters: Clarida, Gali, and Gertler, 2000).

sunspot equilibria. This result is independent of the observability of the steady state, the information imperfection, and the type of a policy rule.

7 Conclusions

This paper has investigated whether stationary sunspot equilibria are stable under adaptive learning when agents' information sets of exogenous variables are imperfect. Sunspot-driven business cycle models are popular to account for non-fundamental macroeconomic fluctuations, while sunspot equilibria are unstable in calibrated business cycle models. The RE literature demonstrates that self-fulfilling fluctuations are driven by the existence of private information of fundamental shocks, while such a relationship has been little analyzed in the framework of adaptive learning. Our paper incorporates such imperfect information into a reduced-form expectational model with the positive feedback of expectations. Imperfect information make agents' forecasting models limited and/or heterogeneous. The paper examines how imperfect information affects the stability of sunspot equilibria under adaptive learning.

We find that when the steady state is observable, stability conditions of sunspot equilibria are unchanged or relaxed by imperfect information. Specifically, as the degrees of limitation and heterogeneity in agents' information sets increase, stability conditions are relaxed. As a result, sunspot equilibria can be stable in standard business cycle models. In a basic New Keynesian model with the observable steady state if agent's information sets are highly imperfect, the violation of the Taylor principle the sufficient and necessary condition for stable sunspot equilibria.

Future works are expected to further clarify the effect of imperfect information on the stability of sunspot equilibria. First, we should analyze models where lagged endogenous variables are included. All of the models in this paper are purely forward-looking to obtain analytical results, but most business cycle models are not only forward-looking, but also backward-looking. Next, our results will be applied to clarifying the mechanism of past macroeconomic fluctuations that were driven by imperfect information. Our results suggest that the imperfection of agents' information sets help the economy stay in self-fulfilling fluctuations. We might be able to find why there happened self-fulfilling fluctuations after innovations of new technology which is familiar for specialists and unfamiliar for the public. Finally, our results also suggest that as the information sets of learning agents reach the perfect (or imperfect) ones, the economy tends to deviate from (converge to) self-fulfilling fluctuations. Thus, our analysis might be able to account for the boom and bust of those fluctuations.

8 Appendix

A Regularity Assumptions

Assumption 2

1. $\det(I_m - B) \neq 0$ *and* $\det(I_{mn} - \Phi \otimes B) \neq 0$.
2. Φ *is a diagonal and nonnegative matrix whose diagonal elements exist in the interval* $[0, 1)$.
3. Γ *is a nonnegative matrix, in which* $0 \leq \rho_{ij} \leq 1$ *for each* $i, j \in \{1, ..., n\}$.

Assumption 2.1 avoids the possibility that a nonexplosive fundamental REE could be indeterminate (see Honkapohja and Mitra, 2006, Proposition 1).

The diagonal representation of Φ in Assumption 2.2 simplifies the analysis by equating the eigenvalues of Φ with its diagonal elements existing in the interval $[0, 1)$. Note that this assumption is not crucial for our analysis, because even if Φ were originally nondiagonal, Eq. (1) could be transformed to an equation that includes a diagonal autocorrelation matrix by premultiplying Eq. (1) by the $n \times n$ matrix formed from the eigenvectors of Φ. The diagonal elements in the interval $[0, 1)$ ensure the stationarity of w_t.

Neither is Assumption 2.3 crucial for our analysis, because any linear model can be transformed to the system with $\Gamma \geq 0$. For example, if any ρ_{ij} is negative in an original model, this negative correlation can be transformed to be positive by changing the sign of w_i (or w_j) and redefining the correlation between $-w_i$ and w_j as $\rho_{ij} \geq 0$. Applying this transformation to all negative correlations, the original model is transformed to the system with $\Gamma \geq 0$.

B Derivation of ODE under HM learning

The ODE under HM learning is obtained by accommodating the global convergence of the ODE associated with an RPE in Evans and Honkapohja (2001, Section 13.1.1). Agent i for each $i \in \{1,\ldots,n\}$ forms $E^*_{it}y_{t+1}$ by using real-time learning with the PLM (16) and the information set $\{y_s, w_{is}, d_s\}_{s=1}^{t}$. We assume the *t-dating of expectations* considered by Evans and Honkapohja (2001, chapter 10): coefficient parameters ϕ_{it} at time t are estimated with past data up to time $t-1$, $\{y_s, w_{is}, d_s\}_{s=1}^{t-1}$, and $E^*_{it}y_{t+1}$ is formed with ϕ_{it} and the contemporaneous data $\{y_t, w_{it}, d_t\}$. The estimates of the coefficient parameters $\phi'_{it} = (a_{it}, c_{it}, d_{it})$ are given by the least-squares projection of y_{t-1} on $z'_{i,t-1} = (1, w_{i,t-1}, \xi_{t-1})$: $Ez_{i,t-1}\left(y_{t-1} - \phi'_{it}z_{i,t-1}\right)' = 0$. Then, the updating rule of ϕ_{it} is shown by the RLS representation:

$$
\begin{aligned}
\phi_{it} &= \phi_{i,t-1} + t^{-1}R_{it}^{-1}z_{i,t-1}\left(y_{t-1} - \phi'_{i,t-1}z_{i,t-1}\right)', && \text{(B.1)}\\
R_{it} &= R_{i,t-1} + t^{-1}\left(z_{i,t-1}z'_{i,t-1} - R_{i,t-1}\right), && \text{(B.2)}
\end{aligned}
$$

where $R_{it} = t^{-1}\sum_{s=1}^{t} z_{i,s-1}z'_{i,s-1}$, which is the updating of the matrix of the second moment of z_{it}.

The stochastic recursive algorithm (SRA) for ϕ_{it} for each i is obtained by substituting the ALM (19) into Eq. (B.1):

$$
\phi_{it} = \phi_{i,t-1} + t^{-1}R_{it}^{-1}z_{i,t-1}\left(\begin{pmatrix} 1 & w'_{t-1} & \xi_{t-1}\end{pmatrix}\begin{pmatrix} S_{a,t-1} & S_{c,t-1} & d_{t-1}\end{pmatrix}' - z'_{i,t-1}\phi_{i,t-1}\right),
$$

where we denote $S_{at} \equiv \beta a_t$, $a_t \equiv \frac{1}{n}\sum_{i=1}^{n} a_{it}$ as the average of the constant term vectors for all types, $S_{ct} \equiv \beta c_t \Phi + \gamma$, $c_t \equiv \frac{1}{n}(c_{1t}, ..., c_{nt})$, $d_t \equiv \frac{1}{n}\sum_{i=1}^{n} d_{it}$.

To obtain the ODEs for ϕ_i associated with the SRA, we have to calculate the unconditional expectations of the updating terms in the SRA. The convergence of the SRA is analyzed by Marcet and Sargent (1989b) in the stochastic approximation approach, which is also introduced by Evans and Honkapohja (2001, chapter 6). Denote the operator E as the expectation of variables for ϕ_i fixed, taken over the invariant distributions of w_t. Then, by letting $Ez_iz'_j = \lim_{t\to\infty} Ez_{it}z'_{jt}$ for each $i, j \in \{1, ..., n\}$, the unconditional expectation of the updating term in Eq. (B.1) is transformed to

$$
\begin{aligned}
&E\left[R_{it}^{-1}z_{i,t-1}\left(\begin{pmatrix} 1 & w'_{t-1} & \xi_{t-1}\end{pmatrix}\begin{pmatrix} S_{a,t-1} & S_{c,t-1} & d_{t-1}\end{pmatrix}' - z'_{i,t-1}\phi_{i,t-1}\right)\right]\\
&\quad = R_i^{-1}\left(Ez_iz'_i\right)\left[\begin{pmatrix} S_a \\ 0 \\ d\end{pmatrix} + \left(Ez_iz'_i\right)^{-1}\begin{pmatrix} 0 & \cdots & 0 \\ \omega_{i1} & \cdots & \omega_{in} \\ 0 & \cdots & 0\end{pmatrix} S'_c - \phi_{i,t-1}\right],
\end{aligned}
$$

where $S_a = \beta a$, $a \equiv \frac{1}{n}\sum_{i=1}^{n} a_i$, $S_c \equiv \beta c\Phi + \gamma$, $c \equiv \frac{1}{n}(c_1, ..., c_n)$, $d \equiv \frac{1}{n}\sum_{i=1}^{n} d_i$, and $Ez_iz'_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{ii} & 0 \\ 0 & 0 & \zeta\end{pmatrix}$ as $E(w_t\xi_t) = 0$ is assumed. Next, the expectation of the updating term in Eq. (B.2) is given by $Ez_iz_i' - R_i$. Hence, the ODEs for ϕ_i and R_i associated with the SRA are obtained as

$$\begin{aligned}\frac{d\phi_i}{d\tau} &= R_i^{-1}\left(Ez_iz_i'\right)\left(T\left(\phi_i\right)' - \phi_i'\right)', &\text{(B.3)}\\ \frac{dR_i}{d\tau} &= Ez_iz_i' - R_i, &\text{(B.4)}\end{aligned}$$

where

$$T\left(\phi_i\right)' \equiv \begin{pmatrix} S_a & \omega_{ii}^{-1}\begin{pmatrix}\omega_{i1} & \cdots & \omega_{in}\end{pmatrix}S_c' & d\end{pmatrix}.$$

A scalar ω_{ij} denotes the covariance of w_i and w_j; $\omega_{ij} \equiv \sigma_{ii}\rho_{ij}\sigma_{jj}$ for each i, j. Furthermore, because R_i and Ez_iz_i' in Eq. (B.4) are asymptotically equal, $R_i^{-1}\left(Ez_iz_i'\right)$ in Eq. (B.3) globally converges to unity. Hence, the stability of the ODE for $\phi_i' = (a_i, c_i, d_i)$ in Eq. (B.3) is determined by smaller differential equations:

$$\frac{d\phi_i}{d\tau} = T\left(\phi_i\right) - \phi_i. \tag{B.5}$$

In the same manner, smaller ODEs for the parameters $\left\{\phi_j\right\}_{j\neq i}^{n}$ are obtained.

The ODEs (B.5) for all i are represented by the ODEs for the aggregate parameters (a, c, d) in Eq. (20) as follows. First, the ODEs for all a_is have the same form, and a is an arithmetic average of all a_is. Then, the convergence property of a is equivalent to that of a_i for each i; the ODEs for all a_is are represented by a single ODE for a that has the same form as that for a_i:

$$\frac{da}{d\tau} = T_a\left(a\right) - a,$$

where $T_a\left(a\right) \equiv S_a$. In the same manner, the ODEs for all d_is are represented by a single ODE for a that has the same form as that for d_i:

$$\frac{dd}{d\tau} = T_d\left(d\right) - d,$$

where $T_d\left(d\right) \equiv d$. Finally, the ODEs for all c_is are represented by a single ODE for the aggregate parameter c. If the ODEs of c_i in Eq.(B.5) for all i are multiplied by $\frac{1}{n}$ and combined in a single $1 \times n$ matrix, the single ODE for c is obtained by:

$$\frac{dc}{d\tau} = T_c\left(c\right) - c,$$

where

$$\begin{aligned}T_c\left(c\right) &\equiv \begin{pmatrix}\frac{1}{n}\omega_{11}^{-1}\begin{pmatrix}\omega_{11} & \cdots & \omega_{1n}\end{pmatrix}S_c' & \cdots & \frac{1}{n}\omega_{nn}^{-1}\begin{pmatrix}\omega_{n1} & \cdots & \omega_{nn}\end{pmatrix}S_c'\end{pmatrix}'\\ &= \left(\beta c\Phi + \gamma\right)\left(\frac{1}{n}\Psi\right),\end{aligned}$$

and

$$
\begin{aligned}
\Psi &\equiv \begin{pmatrix} 1 & \omega_{12}\omega_{22}^{-1} & \cdots & \omega_{1n}\omega_{nn}^{-1} \\ \omega_{21}\omega_{11}^{-1} & 1 & \cdots & \omega_{2n}\omega_{nn}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1}\omega_{11}^{-1} & \omega_{n2}\omega_{22}^{-1} & \cdots & 1 \end{pmatrix} \\
&= diag\left(\sigma_{ii}\right)_{1\leq i\leq n} \cdot \Gamma \cdot diag\left(\sigma_{ii}\right)_{1\leq i\leq n}^{-1}.
\end{aligned}
$$

The derivation is complete.

C Derivation of Proposition 5

Consider the New Keynesian model (33)–(34) and the forward-looking rule (36), and let us obtain the parameter region for matrix B_f to satisfy the positive feedback restriction (29), the real and stationary conditions in Lemma 3, and the stability condition (32).

For B_f to satisfy the positive feedback restriction (29), $\operatorname{tr}\left(-B_f\right) < 0$ and $\det\left(-B_f\right) > 0$ must hold:

$$
\begin{aligned}
\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) &< \alpha^{-1}\left(1+\eta\left(1-\alpha\phi_x\right)\right), \\
\phi_x &< \alpha^{-1}.
\end{aligned}
$$

Next, the stationary condition in Lemma 3 is satisfied if and only if

$$
\begin{gathered}
0 \leq \phi_x < \frac{1+\eta}{\alpha\eta}, \\
\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 0,
\end{gathered}
$$

or

$$
\begin{gathered}
0 \leq \phi_x < \frac{1+\eta}{\alpha\eta}, \\
\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) > \frac{2}{\alpha}\left(1+\eta\left(1-\alpha\phi_x\right)\right),
\end{gathered}
$$

or

$$
\phi_x \geq \frac{1+\eta}{\alpha\eta}.
$$

They are derived from the sufficient and necessary condition of determinate sunspot REEs in Bullard and Mitra (2002, Proposition 4).

Thus, the positive feedback restriction and the stationary condition are summarized as

$$
\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 0, \tag{C.1}
$$

$$
\phi_x < \alpha^{-1}. \tag{C.2}
$$

Note that under Eq. (C.1), the real condition is always satisfied as the eigenvalues of B_f are

$$\frac{1}{2}\left(1+\eta-\alpha\kappa\left(\phi_\pi-1\right)-\alpha\phi_x\right)$$
$$\pm\frac{1}{2}\sqrt{\left(1-\eta+\alpha\kappa\left(\phi_\pi-1\right)+\alpha\phi_x\right)^2-4\alpha\left(\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\eta\right)\right)}.$$

Finally, the stability condition (32) is $\lambda\left[B_f\right]<\Lambda^{-1}$: $\operatorname{tr}\left(B_f-\Lambda^{-1}I_2\right)<0$ and $\det\left(B_f-\Lambda^{-1}I_2\right)>0$. Both conditions provide

$$\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\eta\right) > -\left(1-\Lambda\right)\frac{1-\eta\left(1-\alpha\phi_x\right)\Lambda}{\alpha\Lambda}, \tag{C.3}$$
$$\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\eta\right) > -\frac{1-\eta\left(1-\alpha\phi_x\right)\Lambda}{\alpha\Lambda}-\frac{1-\Lambda}{\alpha\Lambda}. \tag{C.4}$$

Eq. (C.4) is redundant by Eq. (C.3) and $\Lambda<1$.

Combining Eqs. (C.1)–(C.3), the stability conditions of stationary sunspot equilibria of CF representation are given by Eq. (42)–(43). Note that the stability conditions in the univariate version (Proposition 4) correspond to Eq. (42) and $\phi_x=\alpha^{-1}$, and Eq. (43) is obtained.

D Derivation of Proposition 6

Consider the New Keynesian model (33)–(34) and the current-looking rule (35), and let us obtain the parameter region for matrix B_c to satisfy the positive feedback restriction (29), the real and stationary conditions in Lemma 3, and the stability condition (32).

First, the positive feedback restriction (29) provides

$$\operatorname{tr}\left(-B_c\right) = -\frac{\eta+\alpha\kappa+\alpha\eta\phi_x+1}{\alpha\phi_x+\alpha\kappa\phi_\pi+1}<0,$$
$$\det\left(-B_c\right) = \frac{\eta}{\alpha\phi_x+\alpha\kappa\phi_\pi+1}>0,$$

both of which are always satisfied under the sign restrictions of structural parameters in the NK model.

Next, the stationary condition in Lemma 3 is satisfied if and only if

$$\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\eta\right)<0, \tag{D.1}$$

which is derived from the sufficient and necessary condition of determinate sunspot REEs in Bullard and Mitra (2002, Proposition 1). Under Eq. (D.1), the real condition is always satisfied as the eigenvalues of B_c are

$$\frac{1}{2\left(\alpha\phi_x+\alpha\kappa\phi_\pi+1\right)}$$
$$\times\left(\left(\alpha\kappa+\alpha\eta\phi_x+1+\eta\right)\pm\sqrt{\left(\alpha\kappa+\alpha\eta\phi_x+1-\eta\right)^2-4\alpha\eta\left(\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\eta\right)\right)}\right).$$

Finally, the stability condition (32) is $\lambda\left[B_c\right] < \Lambda^{-1}$: $\operatorname{tr}\left(B_c - \Lambda^{-1} I_2\right) < 0$ and $\det\left(B_c - \Lambda^{-1} I_2\right) > 0$. Both conditions provide

$$\begin{aligned}\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) &> -\frac{1}{\alpha}\left(1 - \Lambda\right)\left(1 - \eta\Lambda + \alpha\kappa + \alpha\eta\phi_x\right) \qquad \text{(D.2)}\\ &\quad -\frac{\Lambda}{2\alpha}\left(1 - \eta\Lambda + \alpha\kappa + \alpha\eta\phi_x + \eta\left(1 - \Lambda\right)\right),\\ \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) &> -\frac{1}{\alpha}\left(1 - \Lambda\right)\left(1 - \eta\Lambda + \alpha\kappa + \alpha\eta\phi_x\right). \qquad \text{(D.3)}\end{aligned}$$

Eq. (D.2) is redundant by Eq. (D.3) and $\Lambda < 1$.

Combining Eqs. (D.1) and (D.3), the stability condition of stationary sunspot equilibria of CF representation is given by Eq. (44).

E Derivation of Proposition 7

Consider the New Keynesian model (33)–(34) and the semi-forward-looking rule (37), and let us obtain the parameter region for matrix B_{sf} to satisfy the positive feedback restriction (29), the real and stationary conditions in Lemma 3, and the stability condition (32).

For B_{sf} to satisfy the positive feedback restriction (29), $\operatorname{tr}\left(-B_{sf}\right) < 0$ and $\det\left(-B_{sf}\right) > 0$ must hold:

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) < \alpha^{-1}\left(1 + \eta + \alpha\phi_x\right) \tag{E.1}$$

For B_{sf} to satisfy the stationary condition in Lemma 3,

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) < 0 \text{ or } \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) > 2\alpha^{-1}\left(1 + \eta + \alpha\phi_x\right). \tag{E.2}$$

The derivation of the condition is shown in Appendix F.

Eqs. (E.1)–(E.2) are combined into

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) < 0, \tag{E.3}$$

under which the eigenvalues of matrix B_{sf},

$$\begin{aligned}&\frac{1}{2\alpha\phi_x + 2}\left(\left(1 + \eta + \alpha\phi_x\right) - \alpha\left(\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right)\right)\right)\\ &\pm\frac{1}{2\alpha\phi_x + 2}\sqrt{\left(\left(1 + \eta + \alpha\phi_x\right) - \alpha\left(\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right)\right)\right)^2 - 4\eta\left(1 + \alpha\phi_x\right)},\end{aligned}$$

is always real.

Finally, the stability condition (32) is $\lambda\left[B_{sf}\right] < \Lambda^{-1}$: $\operatorname{tr}\left(B_{sf} - \Lambda^{-1} I_2\right) < 0$ and $\det\left(B_{sf} - \Lambda^{-1} I_2\right) > 0$. Both conditions provide

$$\begin{aligned}\kappa\left(\phi_\pi - 1\right) + \alpha\phi_x\left(1 - \eta\right) &> \alpha^{-1}\left(\eta - \left(\frac{2}{\Lambda} - 1\right)\left(1 + \alpha\phi_x\right)\right), \qquad \text{(E.4)}\\ \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \eta\right) &> -\frac{\left(1 - \Lambda\right)\left(1 - \eta\Lambda + \alpha\phi_x\right)}{\alpha\Lambda} \qquad \text{(E.5)}\end{aligned}$$

Eq. (E.4) is redundant by Eq. (E.5) and $\Lambda < 1$.

Combining Eqs. (E.3) and (E.5), the stability conditions of stationary sunspot equilibria of CF representation are given by Eq. (45).

F Stationary conditions under the semi-forward-looking rule

The sufficient and necessary condition for the stationarity of the ALM (or sunspot ξ_t) is that there exist at least one eigenvalue of B_{sf} that is outside the unit circle. On the other hand, both eigenvalues of B_{sf}^{-1} lie outside the unit circle if and only if

$$\det B_{sf}^{-1} > 1,$$
$$\det B_{sf}^{-1} - \operatorname{tr} B_{sf}^{-1} > -1,$$
$$\det B_{sf}^{-1} + \operatorname{tr} B_{sf}^{-1} > -1;$$

or

$$\begin{aligned} \det B_{sf}^{-1} - \operatorname{tr} B_{sf}^{-1} &< -1, \\ \det B_{sf}^{-1} + \operatorname{tr} B_{sf}^{-1} &< -1. \end{aligned}$$

These conditions are transformed as follows:

$$\phi_x > -\frac{1-\eta}{\alpha},$$
$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) > 0,$$
$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 2\alpha^{-1}\left(1+\eta+\alpha\phi_x\right)$$

or

$$\begin{aligned} \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) &< 0, \\ \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) &> 2\alpha^{-1}\left(1+\eta+\alpha\phi_x\right). \end{aligned}$$

because the determinant and trace of the Jacobian B_{sf}^{-1} are obtained as $\det\left(B_{sf}^{-1}\right) = \frac{1}{\eta}\left(\alpha\phi_x + 1\right)$ and $\operatorname{tr}\left(B_{sf}^{-1}\right) = \frac{1}{\eta} + \frac{1}{\eta}\left(\eta + \alpha\kappa - \alpha\kappa\phi_\pi + \alpha\eta\phi_x\right)$. Thus, the condition of both eigenvalues of B_f^{-1} to lie outside the unit circle is

$$0 < \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 2\alpha^{-1}\left(1+\eta+\alpha\phi_x\right).$$

That is, the sufficient and necessary condition for the stationarity is obtained by

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) < 0 \text{ or } \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\eta\right) > 2\alpha^{-1}\left(1+\eta+\alpha\phi_x\right).$$

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